Range-relaxed Graceful Game*

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Abstract

The range-relaxed graceful game is played in a simple graph G, by two players, Alice and Bob, who alternately assign a previously unused label $f(v) \in \mathcal{L} = \{0, \ldots, k\}$, $k \geq |E(G)|$, to a previously unlabeled vertex $v \in V(G)$. If both ends of an edge $vw \in E(G)$ are already labeled, then the label of the edge is defined as $|f(v) - f(w)|$. Alice's goal is to end up with a vertex labelling of whole G where all of its edges have distinct labels and Bob's goal is to prevent it from happening. When $k = |E(G)|$ the game is called graceful game. Both games were proposed by Z. Tuza in 2017. In this work, we investigate the graceful game for some cartesian products of graphs and corona products of graphs and determine that Bob has a winning strategy in all investigated families independently of who starts the game. Additionally, we also investigate the range-relaxed graceful game and prove that Alice wins the range-relaxed graceful game on any simple graph G with order n for any set $\mathcal{L} = \{0, 1, \ldots, k\}$ with $|\mathcal{L}| \geq \Delta(G)\binom{n-1}{2} + (\Delta(G)^2 + 1)n - \Delta(G)^2$.

Keywords : *graceful labeling, graceful game, range-relaxed graceful game*

1 Introduction

Graph labelling is an area of graph theory whose main concern consists in determining the feasibility of assigning labels to the elements of a graph satisfying certain conditions. In the last decades, many optimization problems have been posed where it is required to label the vertices or the edges of a given graph with numbers. Most of these problems [7–9] emerged naturally from modeling of optimization problems on networks and one of the most investigated is the problem of determining the gracefulness of a graph, proposed by S. Golomb [7] in 1972.

Formally, given a graph $G = (V(G), E(G))$ and a set $\mathcal{L} \subset \mathbb{Z}$, a *labeling* of G is a vertex labeling $f: V(G) \to \mathcal{L}$ that induces an edge labeling $g: E(G) \to \mathbb{Z}$ in the following way: $g(uv)$ is a function of $f(u)$ and $f(v)$, for all $uv \in E(G)$, and g respects some specified restrictions.

Given a graph G and the set of consecutive integer labels $\mathcal{L} = \{0, \ldots, k\}$, $k \geq |E(G)|$, a labelling $f: V(G) \to \mathcal{L}$ is *graceful* if: (i) $k = |E(G)|$; (ii) f is injective; and (iii) if each edge $uv \in E(G)$ is assigned the *(induced) label* $g(uv) = |f(u) - f(v)|$, then all induced edge labels are distinct. When condition (i) in the above definition is relaxed so as to allow $k > |E(G)|$, f is said to be a *range-relaxed graceful labeling* (RRG labeling). The least k needed for G to have a labelling f satisfying conditions (ii) and (iii) in the above definition is called the *gracefulness* of G and is denoted by $grac(G)$. It is known that the parameter $grac(G)$ is defined for every simple graph G [7]. However, $grac(G)$ is not yet determined even for classic families of graphs such as complete graphs [12]. Graceful labelings were introduced by A. Rosa [14] in 1996 and were later so named by S. Golomb [7] who also introduced the range-relaxed variation. Range-relaxed graceful labelings were later investigated by other authors [1, 2].

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From the literature of Graph Labeling [6], it is notorious that labeling problems are usually studied from the perspective of determining whether a given graph has a required labeling. An alternative perspective is to analyze labeling problems from the point of view of combinatorial games. In most combinatorial games, two players — traditionally called Alice and Bob — alternately select and label vertices or edges (typically one vertex or edge in each step) in a graph G which is completely known for both players. In 2017, Z. Tuza [15] surveyed the area of labeling games and posed new labeling games such as the edge-difference distinguishing game, which, in this work, we call range-relaxed graceful game and he also proposed the graceful game. The graceful game was later studied by Frickes et al. [4], that investigated winning strategies for Alice and Bob in some classic families of graphs such as: complete graphs, paths, cycles, wheels, complete bipartite graphs, caterpillars, prisms, and hypercubes.

In this work, we examine the graceful game and study winning strategies for Alice and Bob in some classes of products of graphs, such as: sunlet graphs, grids, generalized book graphs, toroidal grids and the cartesian product of complete graphs and paths. We also investigate the classes of circular snake graphs and crown graphs. Cartesian products of graphs have an interesting behavior since there are cases where Alice has a winning strategy for graphs G_1 and G_2, but she looses on the cartesian product $G_1 \square G_2$. For example, Alice wins the graceful game on C_3 and on K_1 [4], but she looses on the product $C_3 \odot K_1$. Another example is $K_p \square P_q$, where Alice wins on K_3 and P_2 [4], but she looses on $K_3 \square P_2$. Additionally, we also study the range-relaxed graceful game and present a lower bound on the number of consecutive nonnegative integer labels necessary for Alice to win the range-relaxed graceful game on a simple graph G.

This extended abstract is organized as follows: Section 2 presents auxiliary results and definitions; and Sections 3 and 4 present our results on the graceful game and range-relaxed graceful game.

2 Basic notation and auxiliary lemmas

Before presenting our results, some definitions are needed. The *cartesian product* $G_1 \square G_2$ of two graphs G_1 and G_2 is a the graph with vertex set $V(G_1 \square G_2) = V(G_1) \times V(G_2)$ such that $(u_1, v_1)(u_2, v_2) \in E(G_1 \square G_2)$ if and only if either (1) $u_1u_2 \in E(G_1)$ and $v_1 = v_2$, or (2) $v_1v_2 \in E(G_2)$ and $u_1 = u_2$. Another product of graphs is the corona of two graphs, introduced by Frucht and Harary in 1970 [13]. Formally, given a graph G with p vertices and a graph H, the *corona* of G and H, denoted by $G \odot H$, is the graph obtained from G and p copies of H by joining each vertex of G to every vertex of its respective copy of H.

In 2017, Z. Tuza [15] proposed the following maker-breaker game inspired on RRG labelings: given a simple graph G, Alice and Bob alternately assign a previously unused label $f(v) \in \mathcal{L} = \{0, \ldots, k\}$, $k \in \mathbb{N}$, to a previously unlabeled vertex $v \in V(G)$. Alice starts the game. If both ends of an edge $vw \in E(G)$ are already labeled, then the *label* of the edge is defined as $g(vw) = |f(v) - f(w)|$. We say that a move is *legal* if, after it, all edge labels are distinct. The game ends if there is no legal move possible or an RRG labeling is created. Alice *wins* if an RRG labeling of G is created, otherwise Bob *wins*. Tuza called such a game *edge-difference distinguishing*. However, we call it a *range-relaxed graceful game* in order to match with the range-relaxed graceful labeling nomenclature previously used in the literature. Note that, for the case where $|\mathcal{L}| = |E(G)| + 1$, Alice's goal is to end up with a graceful labeling of G. In such a case the game is called *graceful game*.

Next, we state two auxiliary lemmas that are used in our proofs related to the graceful game on some products of graphs.

Lemma 1 (Frickes et al. [4]) *Let G be a simple graph. In any step of the graceful game, Alice can only use the label* 0 *(resp. m) to label a vertex $v \in V(G)$ if v is adjacent to every remaining vertex not yet labeled or v is adjacent to a vertex already labeled by Bob with m (resp.* 0*).* $\square$

Lemma 2 (Frickes et al. [4]) *Let G be a simple graph. If Bob assigns* 0 *(resp.* m*) to a vertex $v \in V(G)$, where v has two non-adjacent vertices or only one adjacent vertex, then Alice is forced to label a vertex adjacent to v with m (resp.* 0*).* □

3 Results on the graceful game

In this section, we state our results on the graceful game of some classes of products of graphs. The *n-sunlet graph*, or simply *sunlet graph*, is the corona of a cycle C_n, $n \geq 3$, with complete graph K_1, denoted $C_n \odot K_1$. R. Frutch [5] proved that all sunlet graphs are graceful. In this work, we characterize the graceful game for all sunlet graphs.

Theorem 3 *Bob has a winning strategy for every sunlet graph $C_n \odot K_1$, $n \geq 3$.*

Sketch of the proof : Consider $G \cong C_n \odot K_1$, $n \geq 3$. First, consider that Bob starts the game by assigning label 0 to a pendant vertex $u \in V(G)$. This forces Alice to assign m to the unique neighbour of u (Lemma 2). Next, Bob assigns 1 to another pendant vertex of G, thus exhausting Alice's possibilities of generating the edge label $m-1$. Therefore, Bob wins the game. Now, consider that Alice starts the game. First, consider that she labels a pendant vertex u with label $a \in \{1, \ldots, m-1\}$ (Lemma 1). Then, Bob assigns 0 to another pendant vertex v. In the next move, Alice is forced to assign label m to a vertex $w \in N(v)$. If $a = 1$, then it is not possible to create the edge label $m-1$ anymore. However, if $a \neq 1$, in the next move, Bob assigns label 1 to an unlabeled pendant vertex of G (such a vertex exists since $n \geq 3$), cancelling Alice's possibilities of creating edge label $m-1$. In both cases, Bob wins the game. In order to conclude the proof, it remains to analyze the case where Alice starts the game playing on a vertex $u \in V(G)$ with degree three. Such a case can be solved similarly to the previous one, by considering separately the cases where she assigns to u a label $a \in \{2, \ldots, m-1\}$ or $a = 1$. In any of these cases, Bob wins the graceful game. □

A *grid* is a simple graph, $P_r \Box P_s$, obtained from the cartesian product of two path graphs P_r and P_s, with $r, s \in \mathbb{N}$ and $r, s \geq 2$. D. S. Jungreis and M. Reid [10] proved that all grids are graceful. Theorem 4 states that Bob wins the graceful game on all grids.

Theorem 4 *Bob has a winning strategy for every grid graph $P_r \Box P_s$, for $r, s \geq 2$.* □

The *generalized book*, $B_{q,r}$, is the graph obtained from the cartesian product of a path P_q with a star S_r, where q is the number of vertices of the path and r is the number of edges of the star. Some results on the gracefulness of generalized books are known [3, 11]. Theorem 5 characterizes the graceful game for all generalized book graphs.

Theorem 5 *Bob wins the graceful game on all generalized books $B_{q,r}$, for $q \geq 2$ and $r \geq 1$.* □

The *toroidal grid* graph $T_{p,q}$, with $p, q \in \mathbb{N}$ and $p, q \geq 3$, is defined as the cartesian product $C_p \Box C_q$ of two cycles C_p and C_q. D. S. Jungreis and M. Reid [10] proved that the toroidal grids $T_{p,q}$ with $p \equiv 0 \pmod 4$ and $q \equiv 0 \pmod 2$ are graceful. Additionally, they also showed that all toroidal grids $T_{p,q}$ with p and q odd are not graceful. The (un)gracefulness of the remaining toroidal grids is still unkown. Theorem 6 characterizes the graceful game for all toroidal grids.

Theorem 6 *Bob has a winning strategy for all toroidal grids.* □

Another class investigated in this work is the cartesian product of a path P_r and a complete graph K_s, for integers $r, s \geq 2$. Although the (un)gracefulness of graphs $K_p \Box P_q$ is largely not settled, the graceful game on this family is characterized as follows.

Theorem 7 *Bob has a winning strategy for every graph $K_p \Box P_q$, for $p, q \geq 2$.* □

In this work, the graceful game was also investigated for two other classes of graphs that are not obtained by graph products: the crown graphs and the (k, n)-circular snakes. A *crown graph*, denoted by R_{2p}, is the bipartite graph on $2p$ vertices, $p \geq 3$, obtained from a complete bipartite graph $K_{p,p}$ by deleting from $K_{p,p}$ the edges of a perfect matching. A (k, n)-*circular snake*, or kC_n-snake, is a connected simple graph with k blocks whose block-cutpoint graph is a path and each of the k blocks is isomorphic to a cycle on n vertices. In this work, it is also shown that Bob wins the graceful game on all members of these families of graphs.

Theorem 8 *Bob wins the graceful game on all crown graphs.* □

Theorem 9 *Bob wins the graceful game on all kC_n-snakes, for $k \geq 2$ and $n \geq 3$.* □

4 Results on the range-relaxed graceful game

In his seminal paper, Z. Tuza [15] asked the following question regarding the range-relaxed graceful game: given a simple graph G and a set of consecutive nonnegative integer labels $\mathcal{L} = \{0, \ldots, k\}$, for which values of k can Alice win the range-relaxed graceful game?

The next two results partially answer Tuza's question by presenting a lower bound for the value of k, for any simple graph and also specifically for trees.

Theorem 10 *Let G be a simple graph on n vertices. Alice wins the range-relaxed graceful game on G for any set $\mathcal{L} = \{0, 1, \ldots, k\}$ with $|\mathcal{L}| \geq \Delta(G)\binom{n-1}{2} + (\Delta(G)^2 + 1)n - \Delta(G)^2$.* □

Theorem 11 *Let T be a tree of order n and maximum degree Δ. Alice wins the range-relaxed graceful game on T for any set $\mathcal{L} = \{0, 1, \ldots, k\}$ with $|\mathcal{L}| \geq (\Delta^2 + \Delta + 1)n - \Delta^2 - 2\Delta$.* □

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