The monochromatic transversal game on clique-hypergraphs of powers of cycles*

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Abstract

We introduce the monochromatic transversal game where the players, Alice and Bob, alternately colours vertices of a hypergraph. Alice, who colours the vertices with red, wins the game if she obtains a red transversal; and Bob wins if he does not let it happen, i.e. there exists a monochromatic blue hyperedge. Both players are enabled to start the game and they play optimally. We analyze the game played on clique-hypergraphs of complete graphs, paths and powers of cycles. For each of these graphs we show a strategy that allows one of the players to win the game.

Keywords : *Combinatorial games, hypergraphs, powers of cycles, transversal.*

1 Introduction

The *clique-hypergraph* $\mathcal{H}(G)$ of an undirected simple graph $G = (V, E)$ is a pair $\mathcal{H} = (V, \mathcal{E})$ where V is the vertex set of G and where the hyperedge set $\mathcal{E}$ is the set of all maximal *cliques* in G, that is, $\mathcal{E}$ is the set of all maximal subsets of V whose vertices induce a complete graph. Clique-hypergraphs were first introduced by Duffus, Sands, Sauer, and Woodrow in the open-problem section of [5] where the authors asked what is the smallest number of colours needed to colour the vertices of $\mathcal{H}(G)$ such that no pair of adjacent vertices of $\mathcal{H}(G)$ is monochromatic. Campos, Dantas, and Mello [4] answered their question in the case of clique-hypergraphs of powers of cycles. They showed that, for powers of cycles, this number is equal to 2, except for odd cycles of size at least 5 where the answer is 3. Bacsó, Gravier, Gyárfás, Preissmann and Sebő [1] proved that this number is 3 for almost all perfect graphs, and Gravier, Hoàng and Maffray [6] studied this subject for graphs with no long path.

A *transversal* in a hypergraph $\mathcal{H}$ is a subset of vertices in V that has a nonempty intersection with every hyperedge of $\mathcal{H}$. This concept has been used in many problems of graph theory and some of them are related to combinatorial games played on hypergraphs. As an example we refer to the *transversal game* introduced by Bujtás, Henning and Tuza [2, 3]. The transversal game played on $\mathcal{H}$ involves of two players, Edge-hitter and Staller, who take turns choosing a vertex from $\mathcal{H}$. Each chosen vertex must hit at least one edge not hit by the vertices previously chosen. The game ends when the set of chosen vertices becomes a transversal in $\mathcal{H}$. Edge-hitter wishes to minimize the number of chosen vertices, while Staller wishes to maximize it.

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In this work, we consider what we call the *monochromatic transversal game*. In this avoider-enforcer game, two players alternately colour the vertices of a hypergraph $\mathcal{H}$. Player 1, who we call Alice, tries to obtain a monochromatic transversal in graph. If she does then she wins the game. Thus, player 2, who we call Bob, tries prevent it from happening, that is, Bob tries to obtain a monochromatic blue hyperedge. The monochromatic transversal game has the following property which we remark below:

Remark 1 *If there exists a strategy that allows Alice (resp. Bob) to win when Bob (resp. Alice) starts the game, then there exists a strategy that allows Alice (resp. Bob) to win when she (resp. he) starts the game.*

We consider the game played on clique-hypergraphs of complete graphs, paths and powers of cycles, showing for each graph, which plays has advantage in the game. Namely, in each graph G we show that there exists a strategy that allows some player to win the game played on that clique-hypergraph $\mathcal{H}(G)$. We label the vertices of V as $\{v_0, \ldots, v_{n-1}\}$ where for any $i < j$ vertices v_i, v_j of G are adjacent (in G) if:

- $G = K_n$ is the complete graph on n vertices;
- $G = P_n$ is a path of length n and $j = i + 1$;
- $G = C_n^k$ is the k-th power of a cycle of length n and $j = i \pm \ell \,(\mathrm{mod}\ n)$, $\ell \leq k$.

The paper is organized as follows: in Section 2 we illustrate the simplest and basic strategies that can be used in the game, where we consider the game being played on the complete graph on n vertices (where Alice always has advantage), on the cycles of length n (where Bob has advantage if ≥ 4) and on on the paths of length n (where Bob has advantage if ≥ 6). In Section 3 we write a complete analysis of the game played on second powers of cycles. We show that there exists a strategy that allows Alice to win the game played on $\mathcal{H}(C_n^2)$, independently of who starts the game, except if $n = 6$. Last, we use algebraic arguments to extend the results of Section 3 for larger powers of cycles.

2 Warming up with clique-hypergraphs of complete graphs, cycles and paths

In order to illustrate the game and its simplest results we start playing it on clique-hypergraphs of complete graphs, cycles and paths. We start with the clique-hypergraph of the complete graph on n vertices K_n.

Proposition 1 *If K_n is a complete graph with $n \geq 2$, then there exists a strategy that allows Alice to win the game played in the clique-hypergraph $\mathcal{H}(K_n)$.*

Proof : The clique-hypergraph $\mathcal{H}(K_n)$ of the complete graph on n vertices contains a unique hyperedge with n vertices. Therefore, by Remark 1, independent of who starts the game, Alice always wins since she obtains a red transversal in her first turn. □

Now we analyze the game played on the clique-hypergraph of a cycle. Since C_3 is isomorphic to K_3 (whose result is contained in Proposition 1), now we consider $n \geq 4$.

Theorem 1 *Let $n \geq 4$. Let C_n denote a cycle of length n. There exists a strategy that allows Bob to win the game on the clique-hypergraph $\mathcal{H}(C_n)$, independently of who starts playing the game.*

Proof : By Remark 1 we may assume that Alice starts the game. Let v_j denote the vertex that is coloured red in Alice's first turn. If Bob colours blue a vertex v_k that is not adjacent to v_j then, independently of which vertex Alice colours red in her next move, there will be an uncoloured vertex v_ℓ, $\ell \in \{k-1 \pmod{n}, k+1 \pmod{n}\}$, that is adjacent to v_k in Bob's next turn. Colouring v_ℓ with blue, Bob obtains his monochromatic blue hyperedge $\{v_k, v_l\}$ and wins the game. $\square$

We finish this section with the analysis of the game on clique-hypergraphs of paths. Moreover, since P_2 is isomorphic to K_2 (whose result is contained in Proposition 1), now we consider $n \geq 3$. First observe that there exists a strategy that allows who started playing to win the game whenever the considered graph is the clique-hypergraph $\mathcal{H}(P_n)$ for $3 \leq n \leq 5$. Indeed, that player must only start colouring vertex $v_{\lceil \frac{n}{2} \rceil}$ and at its second turn: colour $v_{\lceil \frac{n}{2} \rceil - 1}$ or $v_{\lceil \frac{n}{2} \rceil + 1}$ if it is Bob; colour a vertex that is adjacent to Bob's last coloured vertex if it is Alice. In the case of $\mathcal{H}(P_n)$ $n \geq 6$, an argument analogous to the proof of Theorem 1 shows that there exists a strategy that allows Bob to win the game, independently of who starts playing the game.

3 Analysis of the game on the clique-hypergraph of C_n^2

Observe that whenever $n \leq 5$, graphs C_n^2 are isomorphic to the complete graphs on n vertices K_n whose clique-hypergraphs have been analyzed in Proposition 1. Along this section we deal with the remaining second powers of cycles C_n^2, $n \geq 6$. In [4] the maximal cliques in clique-hypergraphs powers of cycles are classified into two types: an *external clique*, whose vertex set is composed by $k+1$ vertices with consecutive indices $v_x, \ldots, v_{x+k \pmod{n}}$ for some $x \in \mathbb{Z}_n$, and an *internal clique*, whose vertex set contains vertices with non-consecutive indices. In this analysis it becomes necessary to know whether it contains or not internal hyperedges. It can be quickly verified that hypergraphs $\mathcal{H}(C_n^2)$ with $n > 6$ (whose hyperedges are cliques of size 3) do not contain these internal cliques. We start with the analysis of these graphs.

Theorem 2 *Let $n > 6$. There exists a strategy that allows Alice to win the game played in the clique-hypergraph $\mathcal{H}(C_n^2)$, independently of who starts playing the game.*

Proof : Recall that all hyperedges of $\mathcal{H}(C_n^2)$, $n > 6$, are external. In order to obtain his monochromatic hyperedge, Bob must colour blue 3 consecutive vertices. Therefore, Alice wins the game playing according to the following rules which are stated in decreasing importance (she only follows rule (j) if its not possible to follow any rule (i) for $i < j$): (1) she colours vertex v_{k-1} whenever there are two blue vertices $v_{k \pmod{n}}$ and $v_{k+1 \pmod{n}}$; (2) she colours vertex v_k whenever there are two blue vertices $v_{k-1 \pmod{n}}$ and $v_{k+1 \pmod{n}}$; (3) if Bob coloured vertex $v_{k \pmod{n}}$ then Alice colours vertex $v_{k+1 \pmod{n}}$; (4) if Bob coloured vertex $v_{k \pmod{n}}$ then Alice colours vertex $v_{k-1 \pmod{n}}$. $\square$

Now we deal with $\mathcal{H}(C_6^2)$ which is the unique clique-hypergraph of a second power of a cycle that contains internal cliques.

Proposition 2 *There exists a strategy that allows Bob to win the game played in the clique-hypergraph $\mathcal{H}(C_6^2)$, independently of who starts playing the game.*

Proof : Firstly, observe that $\mathcal{H}(C_6^2)$ contains the following hyperedges: $e_1 = \{v_0, v_2, v_4\}$, $e_2 = \{v_1, v_3, v_5\}$, $e_3 = \{v_5, v_0, v_1\}$, $e_4 = \{v_2, v_3, v_4\}$, $e_5 = \{v_4, v_5, v_0\}$, $e_6 = \{v_1, v_2, v_3\}$, $e_7 = \{v_0, v_1, v_2\}$ e $e_8 = \{v_3, v_4, v_5\}$, where e_1 and e_2 are the unique internal maximal cliques.

By Remark 1, we may assume that Alice starts playing the game. Moreover, without loss of generality, let us assume that she colours the vertex v_0 with the colour red at her first turn. Since v_0 is adjacent to every vertex in $V(C_n^2)$ except v_3, it makes v_3 be the vertex whose adjacent hyperedges have no red vertex. Therefore, Bob's best option is to colour v_3 blue.

By the symmetry of C_n^2 Alice has two options for her second turn: she can colour v_1 (or v_5) or v_2 (or v_4). If she colours v_1 (resp. v_5) red then she intersects all hyperedges except e_4 and e_8 (resp. e_4 and e_6). Thus, Bob will colour vertex v_4 making her choose one of the vertices v_2 and v_5 (resp. v_1 and v_4). Therefore, she intersects only one of the hyperedges e_4 and e_8 (resp.

e_4 and e_6). Analogously, if Alice colours v_2 (resp. v_4) red at her second turn, Bob will colour vertex v_5 (resp. v_1) making her colour one of the vertices v_1 and v_4 (resp. v_2 and v_5), which forbids her to intersect both hyperedges e_2 and e_8 (resp. e_2 and e_6). $\square$

4 Extending our results to clique-hypergraphs of C_n^k

In this section we use Theorem 2, which concerns the game played on $\mathcal{H}(C_n^2)$, to extend its validity when the game is played on $\mathcal{H}(C_n^k)$ for $k \geq 3$. Similarly to $\mathcal{H}(C_n^2)$, we note that whenever $k \geq \lfloor \frac{n}{2} \rfloor$, graphs C_n^k are isomorphic to the complete graphs on n vertices K_n whose clique-hypergraphs have been analyzed in Proposition 1. Firstly, observe that if Bob can not colour a clique of size k' while playing the game on a clique-hypergraph $\mathcal{H}(G)$, then he can not colour a clique of size k for $k > k'$. By this observation we obtain the following result:

Lemma 1 *Let $n > 6$ and $2 \leq k' < k$. If there exists a strategy that allows Alice to win the game in $\mathcal{H}(C_n^{k'})$, and $\mathcal{H}(C_n^k)$ does not have any internal hyperedge, then there exists a strategy that allows Alice to win the game in $\mathcal{H}(C_n^k)$.*

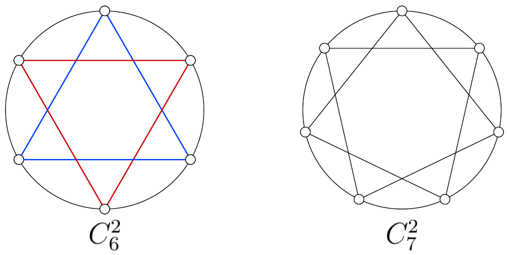

FIG. 1: Graphs C_6^2 (internal maximal cliques in red and blue), and C_7^2 (no internal maximal cliques).

The above lemma allows us to know that Alice has advantage in the game played on clique-hypergraphs of powers of cycles whenever this hypergraphs do not present internal hyperedges. We depict two examples in Figure 1. By Theorem 2 and Lemma 1 we conclude our main result.

Theorem 3 *Let $n > 6$. If $\mathcal{H}(C_n^k)$ has no internal hyperedge then there exists a strategy that allows Alice to win the game played in $\mathcal{H}(C_n^k)$.*

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