Fair allocation of indivisible goods under conflict constraints

Nina Chiarelli[1], Matjaž Krnc[1,2], Martin Milanič[1],
Ulrich Pferschy[3], Nevena Pivač[1], Joachim Schauer[3,4]

[1] University of Primorska, Koper, Slovenia
{nina.chiarelli, matjaz.krnc, martin.milanic}@famnit.upr.si, nevena.pivac@iam.upr.si
[2] University of Ljubljana, Slovenia
[3] University of Graz, Austria
{pferschy, joachim.schauer}@uni-graz.at
[4] FH JOANNEUM, Austria

Abstract

We consider the problem of fairly allocating indivisible items to several agents, each of them having different profit valuations of items. An additional condition allows an incompatibility relation between pairs of items represented by a conflict graph. Hence, every feasible allocation of items to the agents corresponds to a partial coloring, that is, a collection of pairwise disjoint independent sets. The sum of profits of vertices/items assigned to one color/agent should be optimized in a maxi-min sense.

The problem is studied for different classes of conflict graphs. On the one hand we show strong NP-hardness for bipartite graphs and their line graphs, on the other hand we derive pseudo-polynomial solution algorithms for cocomparability graphs and biconvex bipartite graphs.

Keywords : *Fair division, Conflict graph, Partial coloring.*

1 Introduction

The fair allocation of resources to multiple agents is a classical problem in combinatorial optimization. In this work we study the fair allocation of n indivisible goods or items to a set of k agents. Each agent has its own additive utility function over the set of items. The goal is to assign every item to exactly one of the agents such that the minimal utility over all agents is as large as possible. A similar problem is well-known as the *Santa Claus* problem (see [2]), where gifts have to be distributed to kids. It can be also seen as weight partitioning as well as a scheduling problem.

In this contribution we add a *conflict graph* G to the problem. The vertices of the graph correspond to items and an edge between two vertices expresses an incompatibility relation between the two corresponding items, meaning that they may not be allocated to the same agent. This can reflect the fact that items rule out their joint usage or simply the fact that certain items are identical (or from a similar type) and it does not make sense for one agent to receive more than one of these items. Similar conflict graphs were considered for a wide variety of combinatorial optimization problems, e.g., knapsack problem ([11, 12]), bin packing ([9]), scheduling (e.g., [8]) and problems on graphs (e.g., [6]).

Clearly, every feasible allocation to one agent must be an *independent set* in the conflict graph. Therefore, the solution can also be expressed as a *partial k-coloring* of the conflict graph G, but in addition every vertex/item has a profit value for every color/agent and the sum of profits of vertices/items assigned to one color/agent should be optimized in a maxi-min sense.

This problem combines aspects of independent sets, graph coloring, and weight partitioning in an interesting way and offers new perspectives to look at these classical combinatorial optimization problems.

Formally, we consider a set V of items with cardinality $|V| = n$ and k profit functions $p_1, \dots, p_k : V \to \mathbb{Z}_+$. The *satisfaction level* of an ordered k-partition $(X_1, \dots, X_k)$ of V (with respect to $p_1, \dots, p_k$) is defined as the minimum of the resulting profits $p_j(X_j) := \sum_{v \in X_j} p_j(v)$, where $j \in \{1, \dots, k\}$. The classical fair division problem can be stated as follows.

Fair k-Division of Indivisible Goods

Input:	A set V of n items, k profit functions $p_1, \dots, p_k : V \to \mathbb{Z}_+$.
Task:	Compute an ordered k-partition of V with maximum satisfaction level.

For the special case, where all k profit functions are identical, the problem can also be represented in a scheduling setting. There are k identical machines and n jobs, which have to be assigned to the machines by a k-partitioning. The goal is to maximize the minimal completion time (corresponding to the satisfaction level) over all k machines (see [7]). It follows easily from this connection that Fair k-Division of Indivisible Goods is weakly NP-hard for any constant $k \geq 2$ and strongly NP-hard for k being part of the input, even with k identical profit functions.

The first elaborate treatment of Fair k-Division of Indivisible Goods was given in [4], where two approximation algorithms with bounded (but not constant) approximation ratio were given and it was shown that the problem cannot be approximated by a factor better than 1/2. In 2006 Bansal and Sviridenko [2] coined the term *Santa Claus* problem for the case when k is not fixed but part of the input. Since then a huge number of approximation results have appeared on this problem of allocating indivisible goods exploring different concepts of objective functions and various approximation measures.

A different specialization is assumed in the widely studied *Restricted Max-Min Fair Allocation* problem. This is a special case of Fair k-Division of Indivisible Goods where every item $v_i \in V$ has a fixed valuation $p(v_i)$ and every agent either likes or ignores item v_i, i.e., the profit function $p_j(v_i) \in \{0, p(v_i)\}$ (see [1] for a recent overview).

In this contribution we add to the problem a *conflict graph* $G = (V, E)$, where an edge $\{i, j\} \in E$ means that items i and j should not be assigned to the same subset of the partition. Based on the conflict graph we can define a *partial k-coloring* of a graph G as a sequence $(X_1, \dots, X_k)$ of pairwise disjoint independent sets in G (cf. Berge [3]).

Combining the given profit structure with the notion of coloring the vertices of a graph we define for the k profit functions $p_1, \dots, p_k : V \to \mathbb{Z}_+$ and for each partial k-coloring $c = (X_1, \dots, X_k)$ a k-tuple $(p_1(X_1), \dots, p_k(X_k))$, called the *profit profile* of c. The minimum profit of a profile, i.e., $\min_{j=1}^{k}\{p_j(X_j)\}$, is the *satisfaction level* of c and finally implies the definition of our problem:

Fair k-Division Under Conflicts

Input:	A graph $G = (V, E)$, k profit functions $p_1, \dots, p_k : V \to \mathbb{Z}_+$.
Task:	Compute a partial k-coloring of G with maximum satisfaction level.

Note that for $k = 1$, the problem coincides with the strongly NP-hard weighted independent set problem, while the original problem Fair k-Division of Indivisible Goods (arising for an edgeless conflict graph G) is trivial for $k = 1$ and only weakly NP-hard for $k \geq 2$. Thus, the addition of the conflict structure gives rise to a much more difficult problem.

In this contribution we aim at narrowing the gap between tractable cases of Fair k-Division Under Conflicts where the conflict graph permits a pseudo-polynomial solution algorithm, and strongly NP-hard cases. Further results on these aspects will be presented in a future publication currently under preparation.

2 Hardness results for bipartite graphs and their line graphs

Since FAIR k-DIVISION UNDER CONFLICTS is a generalization of the INDEPENDENT SET problem it is interesting to consider graph classes where the latter is still polynomially solvable and investigate whether the former remains strongly NP-hard or not. We start with *bipartite* graphs and show:

Theorem 1 *For each integer $k \geq 2$, the decision version of* FAIR k-DIVISION UNDER CONFLICTS *is strongly NP-complete in the class of bipartite graphs.*

The fairly complicated proof works by a reduction from the CLIQUE problem. In fact, it shows strong NP-hardness for bipartite graphs even for the case when all the profit functions are equal.

Next, we consider the class of *line graphs of bipartite graphs.* Recall that for a given graph G, its line graph has a vertex for each edge of G, with two distinct vertices adjacent in the line graph if and only if the corresponding edges share an endpoint in G. We can show:

Theorem 2 *For each integer $k \geq 2$, the decision version of* FAIR k-DIVISION UNDER CONFLICTS *is strongly NP-complete in the class of line graphs of bipartite graphs.*

The proof is based on a reduction from the following NP-complete problem (see [10]):
Given a bipartite graph and an integer q, does the graph contain a perfect matching and a disjoint matching of size q?

3 Pseudo-polynomial algorithms for special graph classes

After extending the hardness results in Section 2, we also want to identify graph classes which permit positive results. Since the problem is already weakly NP-hard on an edgeless conflict graph, the best we can hope for are pseudo-polynomial algorithms. Since the strong NP-hardness shown in Theorem 1 rules out such a result for FAIR k-DIVISION UNDER CONFLICTS on bipartite graphs, we consider a relevant subclass thereof, namely the class of *biconvex bipartite graphs.*

A bipartite graph $G = (A \cup B, E)$ is biconvex if it has a *biconvex ordering*, that is, an ordering of A and B such that for every vertex $a \in A$ (resp. $b \in B$) the neighborhood $N(a)$ (resp. $N(b)$) is a consecutive interval in the ordering of B (resp. ordering of A).

First of all, we develop a solution for the class of *cocomparability graphs.* A graph $G = (V, E)$ is a *comparability graph* if it has a transitive orientation, that is, if each of the edges $\{u, v\}$ of G can be replaced by exactly one of the ordered pairs (u, v) and (v, u) so that the resulting set A of directed edges is transitive. A graph G is a *cocomparability graph* if its complement is a comparability graph. Comparability graphs and cocomparability graphs are well-known subclasses of perfect graphs. The class of cocomparability graphs is a common generalization of the classes of interval graphs, permutation graphs, and trapezoid graphs (cf. [5]). Since every bipartite graph is a comparability graph, Theorem 1 implies that for each $k \geq 2$, FAIR k-DIVISION UNDER CONFLICTS is strongly NP-complete in the class of comparability graphs. For cocomparability graphs, we managed to construct a highly non-trivial algorithm which permits the following statement.

Theorem 3 *For every $k \geq 1$,* FAIR k-DIVISION UNDER CONFLICTS *is solvable in time $\mathcal{O}(n^{k+2}(Q+1)^k)$ for cocomparability conflict graphs G, where $Q = \max_{1 \leq j \leq k} p_j(V(G))$.*

To tackle the problem on biconvex bipartite conflict graphs G, we use a structural result that allows an ordering of the vertices of G such that the first and last vertices on one side have a special structure and the remaining part of the graph is a *bipartite permutation graph*, i.e., a special case of a cocomparability graph.

Our algorithm enumerates a large, but pseudo-polynomial number of configurations for the first and last vertices on one side and solves for each such configuration the resulting instance on a bipartite permutation graph by the algorithm indicated in Theorem 3. This can be combined to the following result:

Theorem 4 *For every $k \geq 1$,* Fair k-Division Under Conflicts *is solvable in time $\mathcal{O}(n^{3k+2}(Q+1)^k)$ for connected biconvex bipartite conflict graphs G, where $Q = \max_{1\leq j\leq k} p_j(V(G))$.*

Acknowledgements.

The work of this paper was done in the framework of a bilateral project between University of Graz and University of Primorska, financed by the OeAD (SI 22/2018) and the Slovenian Research Agency (BI-AT/18-19-005). The authors acknowledge partial support of the Slovenian Research Agency (I0-0035, research programs P1-0285, P1-0383, research projects J1-9110, N1-0102, J1-1692), and by the Field of Excellence "COLIBRI" at the University of Graz.

References

[1] C. Annamalai, C. Kalaitzis, and O. Svensson. Combinatorial algorithm for restricted max-min fair allocation. *ACM Transactions on Algorithms*, 13(3), 2017.

[2] Nikhil Bansal and Maxim Sviridenko. The Santa Claus problem. In *STOC'06: Proceedings of the 38th Annual ACM Symposium on Theory of Computing*, pages 31–40. 2006.

[3] C. Berge. Minimax relations for the partial q-colorings of a graph. *Discrete Mathematics*, 74(1-2):3–14, 1989.

[4] Ivona Bezakova and Varsha Dani. Allocating indivisible goods. *ACM SIGecom Exchanges*, 5(3):11–18, 2005.

[5] Andreas Brandstädt, Van Bang Le, and Jeremy P. Spinrad. *Graph classes: a survey.* SIAM Monographs on Discrete Mathematics and Applications. Society for Industrial and Applied Mathematics (SIAM), 1999.

[6] A. Darmann, U. Pferschy, J. Schauer, and G.J. Woeginger. Paths, trees and matchings under disjunctive constraints. *Discrete Applied Math.*, 159:1726–1735, 2011.

[7] Bryan L. Deuermeyer, Donald K. Friesen, and Langston Michael A. Scheduling to maximize the minimum processor finish time in a multiprocessor system. *SIAM Journal on Algebraic and Discrete Methods*, 3(2):190–196, 1982.

[8] Guy Even, Magnús M. Halldórsson, Lotem Kaplan, and Dana Ron. Scheduling with conflicts: online and offline algorithms. *Journal of Scheduling*, 12(2):199–224, 2009.

[9] A. Muritiba, M. Iori, E. Malaguti, and P. Toth. Algorithms for the bin packing problem with conflicts. *INFORMS Journal on Computing*, 22(3):401–415, 2010.

[10] Dömötör Pálvölgi. Partitioning to three matchings of given size is NP-complete for bipartite graphs. *Acta Universitatis Sapientiae, Informatica*, 6(2):206–209, 2014.

[11] U. Pferschy and J. Schauer. The knapsack problem with conflict graphs. *Journal of Graph Algorithms and Applications*, 13(2):233–249, 2009.

[12] U. Pferschy and J. Schauer. Approximation of knapsack problems with conflict and forcing graphs. *Journal of Combinatorial Optimization*, 33(4):1300–1323, 2017.