Eventual versus Immediate Spreading: a Full Characterization of Graphs with Coinciding Geodetic and Hull numbers in P_3-convexity

Carmen C. Centeno[1] Lucia D. Penso[2]

[1] Departamento de Computação, PUC-Goiás Goiânia, GO, Brazil, cecilia@pucgoias.edu.br

[2] Contact Author, Institute of Optimization and Operations Research, University of Ulm, P.O.Box 89069, Ulm, Germany, lucia.penso@uni-ulm.de

Abstract

In a propagation dynamics on graph G ruled by the P_3-convexity, the hull number $h(G)$ of a connected graph G is the cardinality of a minimum set of vertices $U \subseteq V(G)$ such that the process of iteratively adding to U all vertices of $G \setminus U$ having two neighbors in U eventually generates the whole vertex set $V(G)$, whereas the geodetic number $g(G)$ of G is the cardinality of a minimum subset of $V(G)$ which accomplishes the same but in a single such iteration. We consider the class $\mathcal{H}$ of connected graphs G for which both parameters coincide, that is, $h(G) = g(G)$. Previous work [6], has characterized a subclass of $\mathcal{H}$. This seminal result is here extended, thus completing the complex task of characterizing the whole class $\mathcal{H}$.

Keywords: spreading, dissemination, networks, P_3-convexity, 2-domination number, geodetic number, hull number, characterization

1 Introduction

The geometric idea of convexity in the Euclidean space has long found direct analogies in pure combinatorial structures. Many types of graph convexities modelling spreading dynamics have been considered in the literature, differing in the kinds of paths (as shortest paths and induced paths) or other structures (as stars) that define a convex set. Each of these convexities has encountered a wide range of applications in a variety of areas related to information networks, physical percolation, epidemiology and computational biology [1, 6, 3, 8].

In particular, the P_3-*convexity* (also known as irreversible 2-conversion [7]) plays an important role in describing a model of propagating properties (such as diseases and beliefs) along a network represented by a graph G where an element of the network becomes "infected" (that is, becomes an element holding the considered property) whenever two of its neighbors are already infected. In this context, the P_3-*hull number* $h(G)$ is the minimum number of elements of the network that must be infected initially, so that the process of iteratively spreading the considered property whenever two neighbors have it *eventually* reaches the whole network, while the P_3-*geodetic number* $g(G)$ is the minimum number of elements that must be infected initially so that the whole network becomes infected *in a single propagation step* (that is, each vertex of the graph is either infected initially or adjacent to at least two initially infected vertices) [6]. Unfortunately, determining the P_3-hull number as well as the P_3-geodetic number of an arbitrary graph is a hard task, and efficient algorithms are only known for quite restricted graph classes[4, 7]. For instance, though the P_3-hull number can be determined in polynomial time for chordal, triangle-free graphs with both parameters coinciding, as well as cubic and subcubic graphs [6, 7, 9], it is already APX-hard for bipartite graphs with maximum degree at most 4 and also NP-complete for planar graphs with maximum degree at most 4, whereas the P_3-geodetic number is NP-complete for planar graphs with maximum degree at most 3 though it admits an efficient computation for other restricted graph classes such as trees, co-graphs and classes of grids [4, 5, 9].

More precisely, given a set S of vertices of a finite, undirected and simple graph G, the *interval of S in the convexity of paths of order* 3, known as the P_3 *convexity*, is the set $[S]_3 = S \cup \{u|u$ belongs to a path P_3 of order 3 between two vertices of $S\} = S \cup \{u|u$ has two distinct neighbors in $S\}$. The set S is P_3*-convex* if $S = [S]_3$ and is P_3*-concave* if $V(G) \setminus S$ is P_3-convex. The P_3*-convex hull of* S is the minimum P_3-convex set containing S and it is denoted by $\langle S \rangle_3$. If $\langle S \rangle_3 = V(G)$, then S is a P_3*-hull set*. The minimum size of a P_3-hull set is the P_3*-hull number* $h(G)$ of G. If $[S]_3 = V(G)$, then S is a P_3*-geodetic set*. The minimum size of a P_3-geodetic set is the P_3*-geodetic number* $g_(G)$ of G, also known as *2-domination number* [6].

Since $h(G) \leq g(G)$ holds trivially, the natural problem of knowing which graphs satisfy it with equality has been considered. In [6], the class $\mathcal{H}$ of all such graphs was partially characterized. Here, we present a full characterization of $\mathcal{H}$.

2 Preliminaries

We consider finite, undirected and simple connected graphs, and use standard notation and terminology. For a graph G, the vertex set is denoted $V(G)$ and the edge set is denoted $E(G)$. For a vertex u of a graph G, the open neighborhood of u in G is denoted $N_G(u)$, the closed neighborhood of u in G is denoted $N_G[u] = N_G(u) \cup \{u\}$, and $d_G(u)$ is the degree of u in G.

We start by formulating a number of structural properties of the class $\mathcal{H}$ [6]. Let G be a fixed graph in $\mathcal{H}$. Let W be a geodetic set of G of minimum order and let $B = V(G) \setminus W$. Since every vertex in B has at least two neighbors in W by definition, G has a spanning bipartite subgraph G_0 with bipartition $V(G_0) = W \cup B$ such that every vertex in B has degree exactly 2 in G_0. Finally, let $\mathcal{G}_0$ denote the set of all bipartite graphs G_0 with a fixed bipartition $V(G_0) = B \cup W$ such that every vertex in B has degree exactly 2. Note that $g(G_0) = h(G_0) = |W|$ for all $G_0 \in \mathcal{G}_0$ [6].

We consider four distinct operations that can be applied to a graph $G_0 \in \mathcal{G}_0$, composing four graph classes whose union yields the graph class $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_1' \cup \mathcal{G}_2 \cup \mathcal{G}_3$, exactly as in [6].

- **Operation** $\mathcal{O}_1$

  For $G_0 \in \mathcal{G}_0$ connected, add one arbitrary edge to G_0.

- **Operation** $\mathcal{O}_1'$

  For $G_0 \in \mathcal{G}_0$ connected, select two vertices w_1 and w_2 from W and arbitrarily add new edges between vertices in $\{w_1, w_2\} \cup (N_{G_0}(w_1) \cap N_{G_0}(w_2))$.

- **Operation** $\mathcal{O}_2$

  For $G_0 \in \mathcal{G}_0$ with exactly two components, add one arbitrary edge between vertices in distinct components of G_0.

- **Operation** $\mathcal{O}_3$

  For $G_0 \in \mathcal{G}_0$ with at least three components and at least one cut vertex belonging to B, choose a non-empty subset X of B such that all vertices in X are cut vertices of G_0 and no two vertices in X lie in the same component of G_0. Add arbitrary edges between vertices in X so that X induces a connected subgraph of the resulting graph. For every component C of G_0 that does not contain a vertex from X, add one arbitrary edge between a vertex in C and a vertex in X.

Let $\mathcal{G}_1$ denote the set of graphs that are obtained by applying operation $\mathcal{O}_1$ once to any connected graph G_0 in $\mathcal{G}_0$. Let $\mathcal{G}_1'$ denote the set of graphs that are obtained by applying

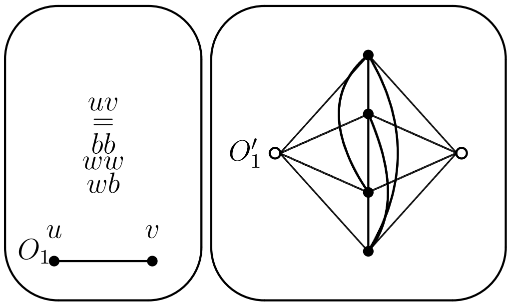

FIG. 1: Operations $\mathcal{O}_1$ and $\mathcal{O}'_1$ for a connected G_0 in $\mathcal{G}_0$.

operation $\mathcal{O}'_1$ once to any connected graph G_0 in $\mathcal{G}_0$. Let $\mathcal{G}_2$ denote the set of graphs that are obtained by applying operation $\mathcal{O}_2$ once to any graph G_0 in $\mathcal{G}_0$ that has exactly two components. Let $\mathcal{G}_3$ denote the set of graphs that are obtained by applying operation $\mathcal{O}_3$ once to any graph G_0 in $\mathcal{G}_0$ that has at least three components as well as at least one cut vertex that belongs to B. Figures 1 and 2 picture $\mathcal{O}_1$, $\mathcal{O}'_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$, where either w or white denotes a vertex in W while either b or black color denotes a vertex in B, of the bipartite $G_0 \in \mathcal{G}_0$ with bipartition $V(G_0) = W \cup B$ where every vertex in B has exactly degree 2. Finally, let $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}'_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$. Note that $\mathcal{G} \subseteq \mathcal{H}$ due to Theorem 10 in [6].

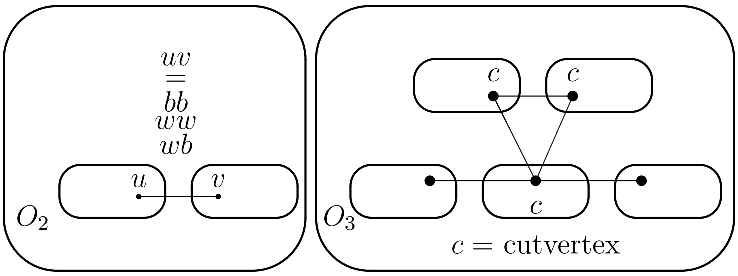

FIG. 2: Operations $\mathcal{O}_2$ for $G_0 \in \mathcal{G}_0$ with 2 components and $\mathcal{O}_3$ for $G_0 \in \mathcal{G}_0$ with at least 3 components.

3 Constructing all graphs in $\mathcal{H}$

Initially construct from $\mathcal{G}_0$ the graph class $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}'_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$, exactly as in [6]. Note that $\mathcal{G}_0$ is contained in $\mathcal{G}'_1$, and thus, in $\mathcal{G} \subseteq \mathcal{H}$ [6]. Now consider the graph class $\mathcal{G}_4$, obtained by applying exactly once operation $\mathcal{O}_4$ to any graph $G \in \mathcal{G}$, where G stems from applying operation $\mathcal{O}_i$ with $i \in \{1, 1', 2, 3\}$ to $G_0 \in \mathcal{G}_0$ with bipartition W and $B = V(G) \setminus W$, where every vertex in B has exactly degree 2 and W is not only both a minimum geodetic set and a minimum hull set of G_0, but also of G due to Theorem 10 in [6].

- Operation $\mathcal{O}_4$. Choose simultaneously (if any) y non-empty disjoint subsets $Y_i \subseteq B, 1 \leq i \leq y$, satisfying

  (i) $d_G(b) = 2 \ \forall b \in Y_i$,

  (ii) there is a unique $w_i \in W$ such that $N_G(b) \cap N_G(b') = \{w_i\} \ \forall b, b' \in Y_i$,

  (iii) if $Y_i \neq Y_{i'}$ then $w_i \neq w_{i'}$,

  (iv) for every C which is a component of $G \setminus \{w_i\}$, $|V(C) \cap Y_i| \subseteq \{0, 1\}$ and the induced subgraph $C^{w_i} = G[V(C) \cup \{w_i\}]$ belongs to $\mathcal{G}_0$ if $|V(C) \cap Y_i| = 1$ and belongs to $\mathcal{G}$ if $|V(C) \cap Y_i| = 0$,

  (v) and for every C^{w_i} with $|V(C^{w_i}) \cap Y_i| = 1$, w_i has degree 1 in C^{w_i} if $G \notin \mathcal{G}_0$.

Now add simultaneously $m_i \geq 0$ edges with both endpoints in Y_i, for a total of $\sum_{i=1}^{y} m_i = m$ new edges.

Roughly speaking, operation $\mathcal{O}_4$ (Figure 3) adds edges between vertices in $Y_i \subseteq B$ which have only two neighbours in W, and where exactly one from both is a common neighbour to all vertices in Y_i, given that certain extra requirements are respected when this single common neighbour in W gets deleted and reinserted. Note that the sets may be placed not only side-by-side (i.e., $x_i \in Y_i$ must cross both w_i and w_j to reach $x_j \in Y_j$ for $1 \leq i, j \leq y$), but also in a top-bottom way (i.e., otherwise), or a mix of both. Intuitively, the nodes in $Y_i \subseteq B$ may be seen as half-brothers from a single common parent whose deletion and reinsertion leaves each half-brother alone in some connected graph of $\mathcal{G}_0$, instead of brothers from two common parents in a connected $G_0 \in \mathcal{G}_0$ as in operation $\mathcal{O}_{1'}$ [6]. In this talk, we show $\mathcal{G} \cup \mathcal{G}_4 = \mathcal{H}$.

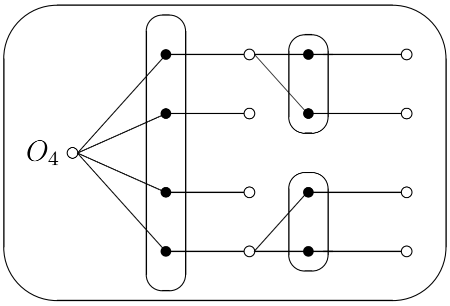

FIG. 3: Three (circled) sets of operation O_4 positioned in a mix of side-by-side and top-bottom way.

References

[1] P. Balister, B. Bollobás, J.R. Johnson, and M. Walters. Random majority percolation. *Random Struct. Algorithms* 36:315–340, 2010.

[2] R.M. Barbosa, E.M.M. Coelho, M.C. Dourado, D. Rautenbach, and J.L. Szwarcfiter. On the Carathéodory Number for the Convexity of Paths of Order Three. *SIAM J. on Disc. Math.* 26:929–939, 2012.

[3] S. Bessy, S. Ehard, L. D. Penso, D. Rautenbach. Dynamic monopolies for interval graphs with bounded thresholds. *Discrete Appl. Math.* 260:256–261, 2019.

[4] C. C. Centeno, M. C. Dourado, J. L. Szwarcfiter. On the Convexity of Paths of Length Two in Undirected Graphs. *Electr. Notes on Disc. Math.* 32: 11–18, 2009.

[5] E.M.M. Coelho, M.C. Dourado and R.M. Sampaio. Inapproximability results for graph convexity parameters. *Theoret. Comput. Sci.* 600:49–58, 2015.

[6] C.C. Centeno, L.D. Penso, D. Rautenbach and V.G. Pereira de Sá. Geodetic number versus hull number in P_3 convexity. *SIAM J. on Disc. Math.* 27(2): 717–731, 2013.

[7] C.C. Centeno, M.C. Dourado, L.D. Penso, D.Rautenbach, and J.L. Szwarcfiter. Irreversible Conversion of Graphs. *Theor. Comput. Sci.* 412: 3693–3700, 2011.

[8] M.C. Dourado, S. Ehard, L.D. Penso and D. Rautenbach. Partial immunization of trees. *To appear in Discrete Optimization*, 2020.

[9] L.D. Penso, F. Protti, D. Rautenbach, and U. dos Santos Souza. Complexity analysis of P3-convexity problems on bounded-degree and planar graphs (and its Corrigendum). *Theor. Comput. Sci.* 607:83–95 (and 704:92–93), 2015 (and 2017).