Hypergraph aspect in equitable colorings of some block graphs*

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Abstract

The problem of equitable edge coloring of hypertrees can be reformulated as equitable vertex coloring of chordal graphs. Motivated by this observation we focus on equitable vertex colorings of chordal graphs. Since the problem is NP-hard in this class, in the paper we consider its restriction to block graphs which form a proper subset of chordal graphs. Recall that an equitable vertex coloring of a graph G is a proper vertex coloring of G such that the sizes of any two color classes differ by at most one. If G is a graph and v is a vertex, then let $\alpha(G,v)$ be the size of a largest independent set of G containing v. Moreover, let $\alpha_{\min}(G) = \min_{v \in V(G)} \alpha(G,v)$. In this paper, we conjecture that for any block graph G the smallest number of colors admitting equitable vertex coloring of G is not greater than $1 + \max\left\{\omega(G), \left\lceil \frac{|V(G)|+1}{\alpha_{min}(G)+1} \right\rceil\right\}$, where $\omega(G)$ is the size of a largest clique of G. The results obtained in the paper support this conjecture. More precisely, we verify it in the class of well-covered block graphs, which are block graphs in which each vertex belongs to a maximum independent set. We also show that the conjecture is true for block graphs with $\alpha_{\min}(G) \leq 2$. In order to derive our results we obtain structural characterizations of the corresponding graphs.

Keywords : *Block-graph, equitable coloring, chromatic spectrum, well-covered block graph, linear hypertree.*

1 Introduction

Let $[k] = \{1, \dots, k\}$. A *k-coloring* of hyperedges of hypergraph $\mathcal{H} = (V, \mathbb{E})$ is a mapping $c : \mathbb{E} \to [k]$ such that no two edges that share a vertex get the same color. An edge k-coloring of $\mathcal{H} = (V, \mathbb{E})$ is *equitable* if each color class is of size $\lceil m/k \rceil$ or $\lfloor m/k \rfloor$. The smallest k such that H admits an equitable edge k-coloring is called the *equitable chromatic index* and is denoted by $\chi'_{=}(\mathcal{H})$.

It is easy to notice that an edge coloring of a hypergraph is equivalent to a vertex coloring of its line graph. A k-coloring of vertices of simple graph $G = (V, E)$ is an assigning colors from the set $[k]$ to vertices in such a way that no two adjacent vertices receive the same color. A vertex k-coloring is *equitable* if each color class is of size $\lceil |V|/k \rceil$ or $\lfloor |V|/k \rfloor$. The smallest k such that G admits an equitable vertex coloring is called the *equitable chromatic number* of G is denoted by $\chi_{=}(G)$. Moreover, note that for a general graph G if it admits an

*The work of the third author has been partially supported by the Italian MIUR PRIN 2017 Project ALGADIMAR "Algorithms, Games, and Digital Markets."

equitable vertex t-coloring it does not imply that it admits an equitable vertex $(t+1)$-coloring (cf. for example $t = 2$ and $G = K_{3,3}$). That is why we also consider the concept of *equitable chromatic spectrum*, i.e. the set of colors admitting equitable vertex coloring of the graph. The smallest k such that G admits an equitable vertex t-coloring for every $t \geq k$ is called the *equitable chromatic threshold* and is denoted by $\chi^*_=(G)$. If $\chi^*_=(G) = \chi_=(G)$ then we say that the equitable chromatic spectrum of G is *gap-free*.

Despite the fact that the corresponding problem for simple graphs has been widely studied, its generalization to hypergraphs does not seem to have been addressed in the literature. To the best of our knowledge there is no paper in the literature that concerns the problem of equitable edge coloring of hypergraphs with the definition given above. Hypergraphs in general are very useful in real-life problems modeling, for example in chemistry, telecommunications, and many other fields of science and engineering. Thus, generalization of equitable coloring of simple graphs to hypergraphs seems to be justified. It is known that the model of equitable coloring of simple graphs has many applications, among others in task scheduling ([5]). Every time when we have to divide a system with binary conflict relations into equal or almost equal conflict-free subsystems we can model this situation by means of equitable graph coloring.

In the paper we have put the question about chordal graphs and their subclasses, and the complexity status of the problem of equitable vertex coloring for them. A graph is *chordal* if every cycle of length at least 4 has a chord. It is also known that a graph G is a chordal graph if and only if it is a line graph of a hypertree, where *hypertree* is defined as a hypergraph that has an underlying tree. Thus equitable edge coloring of hypertrees is equivalent to equitable vertex coloring of chordal graphs. On the other hand, we know (cf. [1], [2]) that the problem of equitable vertex coloring of interval graphs is NP-hard. Since each interval graph is also chordal, we have also NP-hardness of the problem for chordal graphs and in consequence the problem of an equitable edge coloring of hypertrees is also NP-hard. When we take into account the notion of tree-decomposition and classical parameter treewidth of any graph, we may precise the computational complexity of the problem of equitable vertex coloring of some classes of graphs. Bodlaender [1] proved that the problem can be solved in polynomial time for graphs with given tree decomposition and for fixed k. The treewidth of a chordal graph equals the maximum clique size minus one. Bodlaender [1] proved also that the problem of an equitable vertex k-coloring is solvable in polynomial time for graphs with bounded degree even if k is a variable. Thus, the problem of an equitable vertex k-coloring is solvable in polynomial time for chordal graphs with bounded maximum clique size. On the other hand, Gomes et al. [4] proved that, when the treewidth is a parameter to the algorithm, the problem of equitable vertex coloring is W[1]-hard. Thus, it is unlikely that there exists a polynomial time algorithm independent of this parameter. In this paper we address our interest to *block graphs* which are the graphs with every 2-connected component being a clique, where a clique of a graph G is a maximal complete subgraph of G, of size at least two. For block graphs, it is shown in [4] that the problem of equitable vertex coloring is W[1]-hard with respect to the treewidth, diameter and the number of colors. This in particular means that under the standard assumption FPT≠W[1] in parameterized complexity theory, the problem is not likely to be polynomial time solvable in block graphs.

2 Main results

In what follows when we refer to equitable coloring we mean equitable vertex coloring unless stated differently. For a graph G let $\alpha(G)$ be the size of a largest independent set in G, while $\alpha(G, v)$ is the size of a largest independent set that contains the vertex v in G. Define $\alpha_{min}(G)$ as $\min_{v \in V(G)} \alpha(G, v)$. Clearly, $\alpha_{min}(G) \leq \alpha(G)$, and $\alpha_{min}(G) = \alpha(G)$ if and only if every vertex of G lies in a maximum independent set of G. Such graphs are known in the literature as the *well-covered* graphs [6]. For every graph, not necessarily a block graph, it is easy to observe that

$$\chi_=(G) \geq \max\left\{\omega(G), \left\lceil \frac{|V(G)|+1}{\alpha_{min}(G)+1} \right\rceil\right\}. \tag{1}$$

Indeed, the equitable chromatic number of a graph G cannot be less than its clique number as well it cannot be less than $\lceil \frac{|V(G)|+1}{\alpha_{min}(G)+1} \rceil$ what is followed by the assumption that one color is used exactly $\alpha_{\min}(G)$ and any other can be used at most $\alpha_{\min}(G)+1$. It turns out that the number of colors given by the expression on the right side of the inequality is not sufficient to color equitably every block graph. For example, take a clique of size k, $k \geq 2$, and add $k+1$ pendant cliques of size $k+1$ to each vertex (cf. Fig. 1).

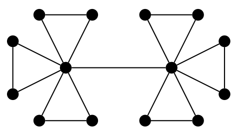

FIG. 1: The $k = 2$ case of the counter-example.

The gap between $\chi_=(G)$ and $\max\left\{\omega(G), \left\lceil \frac{|V(G)|+1}{\alpha_{min}(G)+1} \right\rceil\right\}$ in the above given example is one. This prompted us to the conjecture:

Conjecture 1 *For any block graph G, we have:*

$$\max\left\{\omega(G), \left\lceil \frac{|V(G)|+1}{\alpha_{min}(G)+1} \right\rceil\right\} \leq \chi_=(G) \leq 1 + \max\left\{\omega(G), \left\lceil \frac{|V(G)|+1}{\alpha_{min}(G)+1} \right\rceil\right\}.$$

We have confirmed the conjecture for all block graphs on at most 19 vertices, using a computer. Moreover, the conjecture is true for forests, i.e. for block graphs with $\omega(G) = 2$ [3]. Since the class of connected block graphs in which each cut vertex is on exactly two blocks is equivalent to line graphs of trees, we have $\chi'_=(T) = \chi_=(G)$, for a tree T and its line graph G, $L(T) = G$. Since trees are of Class 1 (i.e. $\chi'_=(T) = \Delta(T)$) and $\Delta(T) = \omega(L(T))$ for a tree T, then we have $\chi_=(G) = \omega(G)$ for connected block graphs in which each cut vertex is on exactly two blocks. Thus our conjecture is true for such block graphs. Moreover, since an arbitrary simple graph G is equitably edge k-colorable for every $k \geq \chi'_=(G)$, then $\chi^*_=(G) = \chi_=(G)$ and the equitable chromatic spectrum of block graph in which each cut vertex is on exactly two blocks is gap-free.
In this paper we prove the conjecture for well-covered block graphs, using unusual tool of Ferrers matrix, as well as for block graphs with small value of $\alpha_{\min}$.

Theorem 1 *Let G be a well-covered block graph. Then G is equitably k-colorable for all $k \geq \omega(G)$.*

Theorem 2 *Conjecture 1 is true for every connected block graph G with $\alpha_{\min}(G) \in \{1, 2\}$.*

Proofs of the above given theorems are preceded by complete characterization of subject subclasses of block graphs.

3 Conclusion and future work

We considered Conjecture 1, which was offering a bound for equitable chromatic number, such that the difference between the upper and lower bounds is at most one. Moreover, both of the bounds are computable in polynomial time. Thus, in some sense, the situation is similar to the chromatic index of graphs, where for simple graphs there is the classical theorem of Vizing and for multigraphs there is the Goldberg conjecture, where a similar gap-one bound is offered for this parameter in the class of all multigraphs. We verified our conjecture for various subclasses of block graphs. Moreover, we gave various examples of block graphs, for which both lower and upper bounds of Conjecture 1 are tight. Usually, when one considers equitable colorings, there are two parameters that one takes into account: the smallest number of colors in an equitable coloring of a graph (equitable chromatic number, $\chi_=(G)$), and the smallest k, such that the graph admits an equitable t-coloring for any $t \geq k$ (equitable chromatic threshold, $\chi^*_=(G)$). As complete bipartite graphs show, these two parameters are not always the same. However,our results confirm our belief that these two parameters have to be the same in the class of block graphs, though we do not have a complete proof of this statement.

One may wonder whether the statement of Conjecture 1 can be extended to arbitrary graphs. In order to see that this extension cannot be true, consider the complete tripartite graph $G = K_{3,5,7}$. What is even more interesting that the conjecture cannot be extended even into the whole class of chordal graphs. We found the smallest chordal graph, for which the

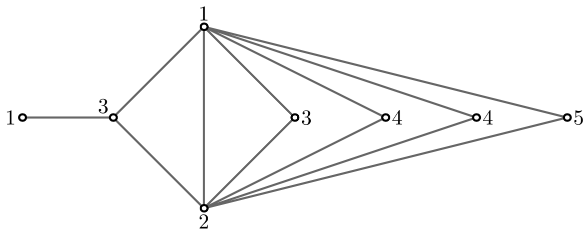

FIG. 2: An example of chordal graph with its equitable coloring.

inequalities form Conjecture 1 do not hold (cf. Fig. 2). Note that in our exemplary graph we have two vertices v_1, v_2, the ones of the highest degree, such that they realise $\alpha_{\min}(G)$, and when we assign color one to vertices from the largest independent set including v_1, the second vertex, v_2 forms maximal independent set, of size 1. So, we have to partition the rest of vertices into minimum number of independent sets of size at most 2. In consequence, by adding vertices of degree 2 that are adjacent to v_1 and v_2, vertices $u_1, \ldots, u_n$, we are able to create arbitrary large chordal graphs with large equitable chromatic number. Thus, the difference between $\chi_=(G)$ and the maximum from Conjecture 1 can be arbitrary large for general chordal graphs.

From our perspective, proving Conjecture 1 and the equality $\chi_=(G) = \chi^*_=(G)$ for block graphs seem promising for future work. It seems also desirable to prove inequalities from Conjecture 1 for other interesting graph classes. In other words, we would like to find other graph classes where the bounds offered by Conjecture 1 are going to hold.

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