# On the Complexity of Minimum Maximal Induced Matching

Juhi Chaudhary*, B S Panda†

Indian Institute of Technology, New Delhi 110016, India

chaudhary.juhi5@gmail.com, bspanda1@gmail.com

**Abstract**

A subset $M \subseteq E_G$ of edges of a graph $G = (V_G, E_G)$ is called a *matching* if no two edges of $M$ share a common vertex. A matching $M$ in $G$ is called an *induced matching* if $G[M]$, the subgraph of $G$ induced by $M$, is same as $G[V(M)]$, the subgraph of $G$ induced by the $M$-saturated vertices of $G$. Max-Ind-Matching is the problem of finding an induced matching of maximum size in a graph. An induced matching $M$ is said to be *maximal* if $M$ is not properly contained in any other induced matching of $G$. Min-Max-Ind-Matching is the problem of finding a maximal induced matching of minimum size. The decision version of this problem is known to be NP-complete for general graphs as well as bipartite graphs [3]. In this paper, we strengthened this result by showing that this problem remains NP-complete for perfect elimination bipartite graphs and dually chordal graphs. On the positive side, we give a linear time algorithm to compute a maximal induced matching of minimum size in cographs. Finally, we show the complexity difference between Max-Ind-Matching and Min-Max-Ind-Matching.

**Keywords** : *Matching, Induced Matching, NP-completeness, Perfect elimination bipartite graphs, Dually chordal graphs, Cographs.*

## 1 Introduction

All graphs considered in this paper are simple, connected, and undirected. Vertices incident to the edges of a matching $M$ are called *$M$-saturated* vertices. A matching $M$ in a graph $G$ is called an *induced matching* if $G[M]$, the subgraph of $G$ induced by $M$, is same as $G[V(M)]$, the subgraph of $G$ induced by the $M$-saturated vertices of $G$. An induced matching $M$ is called maximal if it is not contained in any other induced matching of $G$. Min-Max-Ind-Matching is known to be polynomial time solvable for graph classes like chordal, circular-arc, AT-free graphs [3] and linear time solvable for trees [2]. This problem for a random graph has been studied in [1] and shown NP-complete for bi-size matched graphs in [4]. From the approximation point of view, Min-Max-Ind-Matching cannot be approximated within a ratio of $n^{1-\epsilon}$ for any $\epsilon > 0$, unless $P = NP$ [3].

More formally, the decision version of Max-Ind-Matching and Min-Max-Ind-Matching are defined as follows:

<u>Decide-Max-Ind-Matching:</u>
**Instance:** A graph $G = (V, E)$ and a positive integer $k$.
**Question:** Does there exist an induced matching $M$ in $G$ of size at least $k$?
<u>Decide-Min-Max-Ind-Matching:</u>
**Instance:** A graph $G = (V, E)$ and a positive integer $k$.
**Question:** Does there exist a maximal induced matching $M$ in $G$ of size at most $k$?

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# 2 NP-completeness

## 2.1 Perfect Elimination Bipartite Graphs

An edge $e = xy$ of $G = (X, Y, E)$ is called a *bisimplicial edge* if $N(x) \cup N(y)$ induces a complete bipartite subgraph of $G$. Let $\sigma = (x_1y_1, x_2y_2, \ldots x_ky_k)$ be a sequence of pairwise nonadjacent edges of $G$. Denote $S_j = \{x_1, x_2, \ldots, x_j\} \cup \{y_1, y_2, \ldots, y_j\}$ and let $S_0 = \emptyset$. Then $\sigma$ is called a *perfect edge elimination ordering* for $G$ if each edge $x_{j+1}y_{j+1}$ is bisimplicial in $G_{j+1} = G[(X \cup Y) \setminus S_j]$ for $j = 0, 1, \ldots, k-1$ and $G_{k+1} = G[(X \cup Y) \setminus S_k]$ has no edge. A graph for which there exists a perfect edge elimination ordering is called a *perfect elimination bipartite graph.*

**Theorem 1** Decide-Min-Max-Ind-Matching *is NP-complete for perfect elimination bipartite graphs.*

**Proof :** Clearly, Decide-Min-Max-Ind-Matching is in NP for perfect elimination bipartite graphs. Next, we give a polynomial time reduction from Decide-Min-Max-Ind-Matching for bipartite graphs [3]. Given a bipartite graph $G = (X, Y, E)$, where $X = \{x_1, x_2, ..., x_p\}$ and $Y = \{y_1, y_2, ..., y_l\}$, construct a bipartite graph $G' = (X', Y', E')$ in the following way: For each $y_i \in Y$, add a path $P_i = y_i, a_i, b_i, c_i, d_i, e_i$ of length 5. Formally, $X' = X \cup \{a_i, c_i, e_i \mid 1 \leq i \leq l\}$, $Y' = Y \cup \{b_i, d_i \mid 1 \leq i \leq l\}$ and $E' = E \cup \{y_ia_i, a_ib_i, b_ic_i, c_id_i, d_ie_i, \mid 1 \leq i \leq l\}$. See FIG. 1 for an illustration of the construction of $G'$ from $G$. It is easy to see that $G'$ is a perfect elimination bipartite graph as $(e_1d_1, e_2d_2, ..., e_ld_l, c_1b_1, c_2b_2, ..., c_lb_l, a_1y_1, a_2y_2, ...a_ly_l)$ is a perfect edge elimination ordering of $G'$.

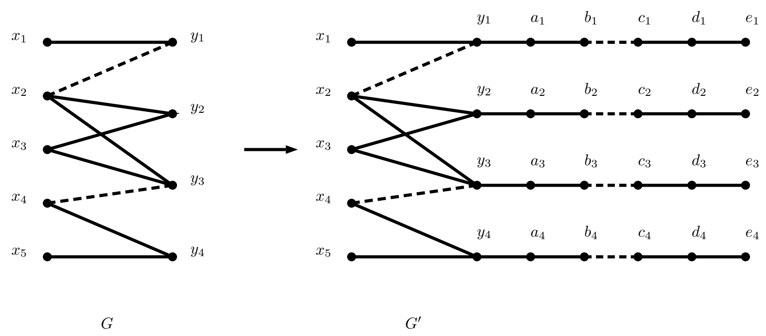


FIG. 1: An illustration of the construction of $G'$ from $G$.

*Claim.* $G$ has a maximal induced matching of size at most $k$ if and only if $G'$ has a maximal induced matching of size at most $k + l$. □

## 2.2 Dually Chordal Graphs

A vertex $u \in N_G[v]$ in a graph $G$ is called a maximum neighbor of $v$ if for all $w \in N_G[v]$, $N_G[w] \subseteq N_G[u]$. An ordering $\alpha = (v_1, v_2, \ldots, v_n)$ of $V_G$ is called a maximum neighborhood ordering, if $v_i$ has a maximum neighbor in $G_i = G[\{v_i, \ldots, v_n\}]$ for all $i, 1 \leq i \leq n$. A graph $G$ is called a *dually chordal graph* if it has a maximum neighborhood ordering.

**Theorem 2** Decide-Min-Max-Ind-Matching *is NP-complete for dually chordal graphs.*

**Proof :** Clearly, Decide-Min-Max-Ind-Matching is in NP for dually chordal graphs. Next, we give a polynomial time reduction from Decide-Min-Max-Ind-Matching for general graphs. Given a graph $G = (V, E)$, where $V = \{v_1, v_2, ..., v_n\}$, construct a dually chordal graph $G' = (V', E')$ in the following way: Take a vertex $v_0$ and make $v_0$ adjacent to all $v_i, 1 \leq i \leq n$. Also, introduce $n$ vertices $w_i$, $1 \leq i \leq n$ and make $v_0$ adjacent to all $w_i's$ also. Then take $n$ copies of $n$ $K_2's$ namely $p_{ij}q_{ij}, 1 \leq i, j \leq n$ and for each $1 \leq i \leq n$ make all $p'_{ij}s$ adjacent to $w_i$ . See FIG. 2 for an illustration of the construction of $G'$ from $G$. It is easy to see that $G'$ is a dually chordal graph as $(q_{11},\dots q_{1n},q_{21},\dots q_{2n},\dots,q_{n1},\dots q_{nn},p_{11},\dots p_{1n},p_{21},\dots p_{2n},$ $\dots,p_{n1},\dots p_{nn},w_1,w_2,\dots,w_n,v_1,v_2,\dots,v_n,v_0)$ is a maximum neighborhood ordering of $G'$.

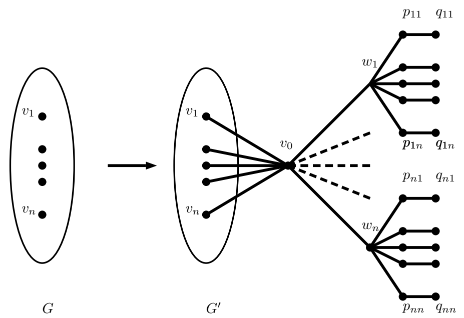


FIG. 2: An illustration of the construction of $G'$ from $G$.

*Claim.* $G$ has a maximal induced matching of size at most $k$ if and only if $G'$ has a maximal induced matching of size at most $k+n$. □

## 3 A linear time algorithm for Cographs

A graph $G$ is called a *complement-reducible graph* or a *cograph* if it can be generated from the single-vertex graph $K_1$ by complementation and disjoint union. It is well known that if a graph $G$ is a cograph, then it is either disconnected or can be written as the join of two cographs $G_1$ and $G_2$.

**Lemma 3.1** *Let $G$ be a cograph such that it is the join of two cographs $G_1$ and $G_2$ and let $M$ be an induced matching in $G$. Then, either $M \subseteq E_{G_1}$ or $M \subseteq E_{G_2}$ or $M$ consists of exactly one edge between a vertex of $G_1$ and a vertex of $G_2$.*

**Theorem 3** *Let $G$ be a cograph such that it is the join of two cographs $G_1$ and $G_2$. Then $M = \{x_1x_2 \mid x_1 \in V_{G_1}, x_2 \in V_{G_2}\}$ is a maximal induced matching in $G$. Furthermore, $M$ is also a minimum maximal induced matching in $G$.*

Next corollary immediately follows by Theorem 3.

**Corollary 1** *The maximal induced matching of minimum size can be computed in $O(n+m)$ time in a cograph.*

## 4 Complexity difference between induced matching and minimum maximal induced matching

In this section, we observe the complexity difference between induced matching and minimum maximal induced matching. We show that Decide-Max-Ind-Matching is NP-complete for $Gx_0$ graphs, but Decide-Min-Max-Ind-Matching is easily solvable for $Gx_0$ graphs. Similarly, Decide-Min-Max-Ind-Matching is NP-complete for $GP_5$ graphs, but Decide-Max-Ind-Matching is easily solvable for $GP_5$ graphs.

Let $\mu_{in}(G)$ denotes the size of a maximum induced matching in $G$ and $\mu'_{in}(G)$ denotes the size of a minimum size maximal induced matching in $G$.

**Definition 1** ($Gx_0$ *graph*). *A bipartite graph $G' = (X', Y', E')$ is called a $Gx_0$ graph if it can be constructed from a bipartite graph $G = (X, Y, E)$ by introducing a new vertex $x_0$ and making $x_0$ adjacent to every $y_i \in Y$.*

**Theorem 4** *If $G'$ is a $Gx_0$ graph, then $\mu'_{in}(G') = 1$.*

**Proof :** Define $M = \{vx_0 \mid v \in V_G\}$. $\square$

**Lemma 4.1** (*see Theorem* 2 *in [5]*) *If $G'$ is a $Gx_0$ graph constructed from a graph $G$ as in Definition* 1*, then $G$ has an induced matching of size at least $k$ if and only if $G'$ has an induced matching of size at least $k$.*

Since DECIDE-MAX-IND-MATCHING is NP-complete for bipartite graphs, the following theorem follows directly from Lemma 4.1.

**Theorem 5** (*see Theorem* 2 *in [5]*) DECIDE-MAX-IND-MATCHING *is NP-complete for $Gx_0$ graphs.*

**Definition 2** (*$GP_5$ graph*). *A bipartite graph $G' = (X', Y', E')$ is called a $GP_5$ graph if it can be constructed from a bipartite graph $G = (X, Y, E)$ by adding a path $P_i = y_i, a_i, b_i, c_i, d_i, e_i$ of length 5 to every $y_i \in Y$. (Note $|Y| = l$)*

**Theorem 6** *If $G'$ is a $GP_5$ graph, then $\mu_{in}(G') = 2l$ .*

**Proof :** Define $M = \{y_i a_i, d_i e_i \mid 1 \leq i \leq l\}$. $\square$

**Lemma 4.2** (*see Theorem 1*) *If $G'$ is a $GP_5$ graph constructed from a graph $G$ as in Definition* 2*, then $G$ has a maximal induced matching of size at most $k$ if and only if $G'$ has a maximal induced matching of size at most $k + l$.*

Since DECIDE-MAX-IND-MATCHING is NP-complete for bipartite graphs, the following theorem follows directly from Lemma 4.2.

**Theorem 7** (*see Theorem 1*) DECIDE-MIN-MAX-IND-MATCHING *is NP-complete for $GP_5$ graphs.*

# 5 Conclusion and Open Problems

Exploring the parameterized complexity of the problem will be interesting.

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