1,2,3 conjecture on some subclasses of bipartite graphs

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Abstract

An edge k- weighting w, $w : E(G) \to [k] := \{1, 2, ..., k\}$, is a proper vertex coloring by sums if $\sum_{e\sim u} w(e) \neq \sum_{e'\sim v} w(e^{'})$ for every $uv \in E(G)$. The least value of k such that G has a edge k-weighting which is a proper vertex coloring by sums is denoted by $\chi^{e}_{\Sigma}(G)$. We focus on the problem of finding vertex coloring 2-edge weighting of bipartite graphs where we consider the set of edge weights as $\{1, 2\}$ and $\{0, 1\}$. We show that vertex coloring 2-edge weighting of chain graph can be computed in linear time for both edge weights $\{1, 2\}$ and $\{0, 1\}$. We also prove that if two graphs have vertex coloring $\{1, 2\}$-edge weightings, then their Cartesian product also has vertex coloring $\{1, 2\}$-edge weighting. Finally, we give some subclasses of bipartite graphs that do not admit vertex coloring $\{0, 1\}$-edge weightings.

Keywords : *Vertex coloring 2-edge weighting, Chain graph, Cartesian product.*

1 Introduction

An edge k-weighting is a function $w : E(G) \to [k] := \{1, 2, ..., k\}$. An edge k- weighting w is a proper vertex coloring by sums if $\sum_{e\sim u} w(e) \neq \sum_{e'\sim v} w(e^{'})$ for every $uv \in E(G)$. We denote the smallest value of k such that a graph G has a edge k-weighting which is a proper vertex coloring by sums by $\chi^{e}_{\Sigma}(G)$. A graph G is nice if no connected component is isomorphic to K_2. In 2004, Karonski, Luczak and Thomason [3] gave the famous 1-2-3 conjecture.

1-2-3 conjecture *If G is nice, then* $\chi^{e}_{\Sigma}(G) \leq 3$.

The motivation for the 1-2-3 Conjecture comes from the study of graph irregularity strength. An edge weighting of a graph G is an irregular assignment if, for any pair of vertices $u, v \in V(G)$, the sum of weights of edges incident to u differs from the sum of weights of edges incident to v. The irregularity strength of a graph G is the smallest value of k such that G has an irregular assignment from $[k]$.

The best known bound for χ^{e}_{Σ} was obtained in 2010 by Kalkowski, Karonski, Pfender [2] as $\chi^{e}_{\Sigma} \leq 5$ for any nice graph G. In [1], Dudek and Wajc showed that determining whether a given graph has vertex coloring 2-edge weighting is NP-complete. In [4], Davoodi et al. studied vertex-coloring edge-weighting of Cartesian product of graphs. Recently, Thomassen [5] characterized completely the bipartite graphs having an edge-weighting with weights $\{1, 2\}$ such that neighboring vertices have distinct weighted degrees. However, the problem is still open if the edge weights are assigned from $\{0, 1\}$.

2 Preliminaries

In this paper, we consider simple and connected graphs. For a graph $G = (V, E)$, the set of neighbors of a vertex v, $\{u \in V(G) : uv \in E(G)\}$, is denoted by $N(v)$. The degree of a vertex v is defined as $d(v) = |N(v)|$. Let n and m denote the number of vertices and number of edges of a graph G, respectively. A graph G is said to be *bipartite* if $V(G)$ can be partitioned into two nonempty disjoint sets X and Y such that every edge of G connects a vertex in X to another vertex in Y. A bipartite graph $G = (X, Y, E)$ is said to be *complete bipartite* if every vertex of X is adjacent to every vertex of Y. A bipartite graph $G = (X, Y, E)$ with $|X| = p$ and $|Y| = q$, is called a *chain graph* if the neighborhoods of the vertices of X form a chain, i.e., the vertices of X can be linearly ordered, say $x_1, x_2, \ldots, x_p$ such that $N(x_1) \subseteq N(x_2) \subseteq \cdots \subseteq N(x_p)$. If $G = (X, Y, E)$ is a chain graph, then the neighborhoods of the vertices of Y also form a chain. An ordering $\sigma = (x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_q)$ of $X \cup Y$ is called a chain ordering if $N(x_1) \subseteq N(x_2) \subseteq \cdots \subseteq N(x_p)$ and $N(y_1) \supseteq N(y_2) \supseteq \cdots \supseteq N(y_q)$. It is known that every chain graph admits a chain ordering that can be computed in linear time. The Cartesian product of two graphs G and H is the graph $G\Box H$ with vertices $V(G\Box H) = V(G) \times V(H)$, and for which $(x, u)(y, v)$ is an edge if $x = y$ and $uv \in E(H)$, or $xy \in E(G)$ and $u = v$.

3 Vertex coloring $\{1, 2\}$-edge weighting

In this section, we discuss vertex coloring $\{1, 2\}$-edge weighting for chain graphs and Cartesian product of graphs.

3.1 Chain Graphs

In this subsection, first we give a linear time algorithm for finding the vertex coloring $\{1, 2\}$-edge weighting, using at most 3 colors, of a complete bipartite graph, which is a proper subclass of chain graphs.

Proposition 1 *If G is a complete bipartite graph, then the vertex coloring $\{1, 2\}$-edge weighting of G can be computed in linear time using at most 3 colors.*

Proof : Let $G = (X, Y, E)$ be a complete bipartite graph. It can be easily seen that if $|X| \neq |Y|$, then assigning weight 1 to each edge gives vertex coloring $\{1, 2\}$-edge weighting of G using 2 colors, namely, $|X|$ and $|Y|$. Now suppose $|X| = |Y| = r$, say. Consider the following edge weighting w: $w(x_i, y_j) = 2$, if (i mod $2 = 0$) and $w(x_i, y_j) = 1$, otherwise, for each $1 \leq j \leq r$. Note that this edge weighting induces a proper vertex coloring by sums since each vertex of X will be assigned color either r or $2r$ and the vertices of Y receive color $r + \lfloor \frac{r}{2} \rfloor$, where $\lfloor . \rfloor$ represents the greatest integer function. □

Theorem 1 *If G is a chain graph, then the vertex coloring $\{1, 2\}$-edge weighting of G can be computed in linear time.*

Proof : Let $G = (X, Y, E)$ be a chain graph. Note that since we have proved the result for complete bipartite graphs in Proposition 1, we may assume that G is not a complete bipartite graph. Since $G = (X, Y, E)$ is a chain graph, there exists a chain ordering, $\sigma = (x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_q)$ of $X \cup Y$ such that $N(x_1) \subseteq N(x_2) \subseteq \cdots \subseteq N(x_p)$ and $N(y_1) \supseteq N(y_2) \supseteq \cdots \supseteq N(y_q)$. Now we consider the following cases:

Case 1 : $|X| = |Y| = 2r$

Consider the edge weights given by w:

$$w(x_i, y_j) = \begin{cases} 1, \text{ if } j = 1 \text{ and } 1 \leq i \leq 2r, \\ 2, \text{ otherwise.} \end{cases}$$

Since y_1 is adjacent to all $2r$ vertices of X and contribute 1 to each vertex of X, the vertices of Y receive even colors and the vertices of X receive odd colors.

Case 2 : $|X| = |Y| = 2r + 1$

Now we further consider different cases depending on the degree of x_1.

Subcase (a): $d(x_1) \leq r$

Consider the edge weights given by w_1:

$$w_1(x_i, y_j) = \begin{cases} 1, \text{ if } j = 1 \text{ and } 2 \leq i \leq 2r + 1, \\ 2, \text{ otherwise.} \end{cases}$$

Since y_1 is adjacent to all $2r + 1$ vertices of X and contribute 1 to each vertex of X except x_1, the vertices of Y receive even colors and the vertices of $X - \{x_1\}$ receive odd colors. Note that x_1 always receives an even color but since $d(x_1) \leq r$, x_1 may only be adjacent to vertices of Y that have the same parity but more value than x_1.

Subcase (b): $d(x_1) \geq r+1$

Suppose $d(x_1) = r+1$. This implies that $\{y_1, \ldots, y_{r+1}\}$ are adjacent to each vertex of X. Consider the edge weights given by w_2:

$$w_2(x_i, y_j) = \begin{cases} 2, \text{ if } 1 \leq j \leq (r+1) \text{ and } 1 \leq i \leq 2r+1, \\ 1, \text{ otherwise.} \end{cases}$$

Note that $\{y_1, \ldots, y_{r+1}\}$ receive color $2(2r+1)$ and $\{y_{r+2}, \ldots, y_{2r+1}\}$ receive color at most $2r+1$. Since the vertices of X receive color at least $2(r+1)$, w_2 is a proper vertex coloring by sums. A similar argument holds for $d(x_1) > r+1$.

Case 3 : If the number of vertices in one of the sets, X or Y, is even.

Suppose $|X| = 2l$, for some positive integer l. Since y_1 is adjacent to all $2l$ vertices of X, the following edge weighting given by w result in odd colors for the vertices of X and even colors for the vertices of Y:

$$w(x_i, y_j) = \begin{cases} 1, \text{ if } j = 1 \text{ and } 1 \leq i \leq 2l, \\ 2, \text{ otherwise.} \end{cases}$$

Similarly, if $|Y| = 2l$, then assign weight 1 to all the vertices adjacent to x_p and weight 2 to the other edges to obtain vertex coloring by sums.

Case 4 : If $|X| = 2r+1$ and $|Y| = 2l+1$, $(l \neq r)$.

Let us assume, without loss of generality, that $r > l$.

Consider the edge weights given by w:

$$w(x_i, y_j) = \begin{cases} 1, \text{ if } i = 2r+1 \text{ and } 2 \leq j \leq 2l+1, \\ 2, \text{ otherwise.} \end{cases}$$

Note that all the vertices of Y except y_1 receive an odd color and each vertex of X receive an even color. Since y_1 is adjacent to each vertex of X and recieves 2 from each edge, y_1 gets color $2(2r+1)$. Now any vertex of X receives color at most $2(2l+1)$ which is strictly less than the color of y_1 as $r > l$.

This completes the proof of the theorem. $\square$

3.2 Cartesian product of graphs

In this subsection, we show that the Cartesian product of graphs G and H, $G \square H$ has a vertex coloring $\{1,2\}$-edge weighting if both G and H have vertex coloring $\{1,2\}$-edge weightings.

Theorem 2 *Let G and H be graphs that have vertex coloring $\{1,2\}$-edge weighting, then $G \square H$ has vertex coloring $\{1,2\}$-edge weighting that can be computed in linear time.*

Proof : Let w_G and w_H be vertex coloring $\{1,2\}$-edge weightings of G and H, respectively. We define w, vertex coloring $\{1,2\}$-edge weighting of $G \square H$ for each edge $(x,u)(y,v)$ as follows:

$$w((x,u)(y,v)) = \begin{cases} w_H(uv), \text{ if } x = y \text{ and } uv \in E(H), \\ w_G(xy), \text{ if } xy \in E(G) \text{ and } u = v. \end{cases}$$

Note that same weight is added to each vertex of each copy of G from the new edges of H and vice versa. Suppose w is not a proper coloring by sums. This implies that there exists an edge $(x,u)(y,v) \in E(G \square H)$ such that its endpoints have same color. Suppose $x = y$ and $uv \in E(H)$. Since $x = y$, the total weights added to (x,u) and (y,u) due to the edges in G are the same. But that would imply that the total weights contributed by the edges in H to u and v are same, this is a contradiction to w_H being a vertex coloring $\{1,2\}$-edge weightings of H. Thus, w is a vertex coloring $\{1,2\}$-edge weighting of $G \square H$.

$\square$

4 Vertex coloring $\{0,1\}$-edge weighting

In this section, we discuss vertex coloring $\{0,1\}$-edge weighting of chain graphs and present some subgraphs of bipartite graphs that never admit vertex coloring $\{0,1\}$-edge weighting.

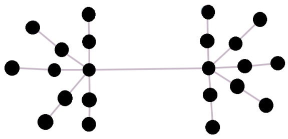

FIG. 1: Graph formed by adding 5 distinct P_3 added to each vertex of K_2.

4.1 Chain graph

In this subsection, we show that vertex coloring $\{0,1\}$-edge weighting using 3 colors of chain graph can be determined in linear time.

Theorem 3 *If G is a chain graph, then the vertex coloring $\{0,1\}$-edge weighting of G can be computed in linear time using 3 colors.*

Proof : Let $G = (X, Y, E)$ be a chain graph. Then there exists a chain ordering of G, $\sigma = (x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_q)$ of $X \cup Y$ such that $N(x_1) \subseteq N(x_2) \subseteq \ldots N(x_p)$ and $N(y_1) \supseteq N(y_2) \supseteq \ldots N(y_q)$. Note that if G is a star graph with more than 1 non pendant vertices, then assigning weight 1 to any two edges gives a vertex coloring $\{0,1\}$-edge weighting of G using 3 colors. Without loss of generality, we assume that $|Y| \geq |X|$ and G is not a star graph. Consider the edge weights given by w:

$$w(x_i, y_j) = \begin{cases} 1, \text{ if } i = p \text{ and } 1 \leq j \leq q, \\ 0, \text{ otherwise.} \end{cases}$$

Note that all the vertices of Y are adjacent to x_p and w assigns color 1 to each vertex of Y. Each vertex of X except x_p receive color 0 and x_p receives color $|Y|$. $\square$

Proposition 2 *If $G = (X, Y, E)$ is a bipartite graph with either $x \in X$ satisfying $d(x) = |Y|$ or $y \in Y$ satisfying $d(y) = |X|$, then applying the above weighting on x or y, respectively, instead of x_p results in a vertex coloring $\{0,1\}$-edge weighting of G using at most 3 colors.*

4.2 Some subclasses of bipartite graphs that do not admit vertex coloring $\{0,1\}$-edge weighting

In this subsection, we give some subclasses of bipartite graphs that do not admit vertex coloring $\{0,1\}$-edge weighting.

1. C_{4r+2} and P_{4r+2}, $r \geq 1$.
2. Graphs formed by adding r distinct P_3 to each vertex of K_2. (See Figure 1.)
3. Graphs formed by adding C_{4r+2} to each end vertex of P_{4r+3}.

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