Tight bounds on global edge and complete alliances in trees

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Abstract

In the talk the authors present some tight upper bounds on global edge alliance number and global complete alliance number of trees. Moreover, we present our NP-completeness results from [8] for global edge alliances and global complete alliances on subcubic bipartite graphs without pendant vertices. We discuss also polynomial time exact algorithms for finding the minimum global edge alliance on trees [7] and complete alliance on trees [8].

Keywords : *Global alliance, global edge alliance, global complete alliance.*

1 Introduction

Let G be a simple nonempty graph. By $G[A]$, where $A \subset V(G)$, we denote a subgraph of G induced by set A. Let $X \subset V(G)$, by an *open neighborhood* of X in graph G we mean the set $\{v \in V(G) \colon \exists_{u \in X}\{v, u\} \in E(G)\}$, denoted by $N_G(X)$. By a *closed neighborhood* of X in graph G we mean set $X \cup N_G(X)$, denoted by $N_G[X]$. Set X is a *dominating set* of G iff $V(G) = N_G[X]$. Set X is a *total dominating set* of G iff $V(G) = N_G(X)$. By an *isolated vertex* (in a graph G) we mean a vertex of degree 0. By a *pendant vertex* we mean a vertex of degree 1. A neighbor of a pendant vertex in a tree we call a *support* vertex.

Let $S \subset V(G)$. For any non-empty subset X of S we define the predicate $SEC_{G,S}(X) = true$ iff $|N_G[X] \cap S| \geq |N_G[X] \setminus S|$. We use the notation $SEC(X)$ instead of $SEC_{G,S}(X)$, if G and S are clearly given. For the sake of notation simplicity, we write $N_G[v]$ and $N_G[v, u]$ instead of $N_G[\{v\}]$ and $N_G[\{v, u\}]$, respectively, and analogously, $SEC(v)$ and $SEC(v, u)$.

The concepts of edge alliances and complete alliances in graphs arise from the the concept of defensive alliances [4, 5]. An *alliance* (or *defensive alliance*) in a graph G is a set $S \subset V(G)$ such that $\forall_{v \in S} SEC_{G,S}(v) = true$. If S is also a dominating set of G, we call it a global alliance of G, and by $\gamma_a(G)$ we denote the size of the minimum global alliance in a graph G. The problem has interesting applications in different fields of study, most notably in web communities [3] and in fault-tolerant computing [9].

An *edge alliance* in a graph G is a set $S \subset V(G)$ such that there is no isolated vertices in $G[S]$ and $\forall_{\{u,v\} \in E(G[S])} SEC_{G,S}(u, v) = true$. If S is also a dominating set of G, we call it a global edge alliance of G, and by $\gamma_{ea}(G)$ we denote the size of the minimum global edge alliance in a graph G. The concept was introduced in [6] and studied in [7], where authors proved $\mathcal{NP}$-completeness of the global edge alliance problem for subcubic graphs, and showed lower bounds on γ_{ea} for arbitrary graphs, and gave exact values of γ_{ea} for some classes of graphs, e.g., for complete multipartite graphs and for complete k-ary trees. Also, authors constructed polynomial time algorithm solving the global edge alliance problem for trees.

We introduce the concept of *complete alliance*. Let $S \subset V(G)$ for a given graph G. Set S is a *complete alliance* in G iff for each clique $K \subset V(G[S])$ we have $SEC_{G,S}(K) = true$. A complete alliance S is a *global complete alliance* in G if it also dominates G. By $\gamma_{ca}(G)$ we denote the size of the minimum global complete alliance in graph G.

In the paper we prove some tight bounds on the global edge alliance and the global complete alliance in trees. Moreover, we prove some NP-completeness results for the above problems for subcubic bipartite graphs with some other restrictions. The polynomial time exact algorithm for finding the minimum global complete alliance is also constructed.

2 Bounds on γ_{ea} and γ_{ca}

In the paper we present new upper bounds on γ_{ea} and γ_{ca} for trees. Let us observe some relations between γ_a, γ_t, γ_{ea} and γ_{ca}. From the definition we have

Proposition 1 *Let G be a connected graph. Then,*

1. $\gamma_t(G) \leq \gamma_{ea}(G)$,
2. $\gamma_a(G) \leq \gamma_{ca}(G)$,
3. *if* $\delta(G) \geq 2$, *then* $\gamma_{ca}(G) \geq \gamma_{ea}(G) \geq \gamma_t(G)$,
4. *if* $\Delta(G) \leq 3$, *then* $\gamma_{ea}(G) \geq \gamma_{ca}(G) \geq \gamma_a(G)$ *and* $\gamma_{ea}(G) \geq \gamma_t(G) \geq \gamma_a(G)$.

FIG. 1: The examples of the global alliance number and the global edge alliance number: (a) $\gamma_a = 2 < \gamma_{ea} = 3$ and (b) $\gamma_{ea} = 3 < \gamma_a = 5$.

There is no such a relation, in general, between the global alliance number and the global edge alliance number, which is shown in Fig. 1. Note that γ_{ca} and γ_{ea} are also not related in the class of trees with the maximum degree bounded by 4. In Fig. 2 there are presented two graphs T_1 and T_2 such that $\gamma_{ca}(T_1) < \gamma_{ea}(T_1)$ and $\gamma_{ea}(T_2) < \gamma_{ca}(T_2)$.

FIG. 2: $\gamma_{ca}(T_1) = 3 < \gamma_{ea}(T_1) = 4$ and $\gamma_{ca}(T_2) = 5 > \gamma_{ea}(T_2) = 3$.

It is worth to observe also that γ_{ca} and γ_t are not related in subcubic trees. In Fig. 3 there are presented two graphs T_1 and T_3 such that $\gamma_{ca}(T_1) < \gamma_t(T_1)$ and $\gamma_{ca}(T_3) > \gamma_t(T_3)$.

FIG. 3: $\gamma_{ca}(T_1) = 3 < \gamma_t(T_1) = 4$ and $\gamma_{ca}(T_3) = 3 > \gamma_t(T_3) = 2$.

In the paper [4] the authors showed the following tight upper bound for trees.

Theorem 1 *[4] Let T be a tree with at least three vertices. Then,* $\gamma_a(T) \leq \frac{3n(T)}{5}$.

We proved similar result for the global edge alliance for trees.

Theorem 2 *Let T be a tree with $n(T) \geq 3$. Then,*

1. $\gamma_{ea}(T) \leq \frac{2n(T)}{3}$,
2. $\gamma_{ea}(T) = \frac{2n(T)}{3}$ *if and only if T is a P_2-corona of a tree.*

Let $s(T)$ be the number of support vertices in a tree T. Some upper bounds (using $n(T)$ and $s(T)$) on the global alliance number and the total domination number in trees were proved in [2] and [1], respectively.

Theorem 3 *[2] Let T be a tree with at least three vertices. Then, $\gamma_a(T) \leq \frac{n(T)+s(T)}{2}$.*

Theorem 4 *[1] Let T be a tree with at least three vertices. Then, $\gamma_t(T) \leq \frac{n(T)+s(T)}{2}$.*

In the paper we present our results that are stronger than the results above.

Theorem 5 *Let T be a tree with $n(T) \geq 3$. Then, $\gamma_{ca}(T) \leq \frac{n(T)+s(T)}{2}$.*

Theorem 6 *Let T be a tree with $n(T) \geq 3$. Then, $\gamma_{ea}(T) \leq \frac{n(T)+s(T)}{2}$.*

Moreover, we prove that these bounds are tight by showing the full characterisation of trees for which the above parameters are equal to the given bound.

Definition 1 *A **star component** S of a tree T is a subgraph of T induced by a set of vertices $N_T[s]$, where $s \in V(T)$ is a support vertex in T. A **pendant star** S in a tree T is a star component of T such that exactly one edge $e \in E(T)$ connects S with the subgraph $T[V(T) \setminus V(S)]$. If $v \in e, v \in V(T) \setminus V(S)$ is a leaf in $T[V(T) \setminus V(S)]$, then S is a **leaf-pendant star**.*

Definition 2 *Let S be a star component of a tree T and let $s \in V(S)$ be a support vertex in S. Let $l = |N_S(s)|$. Let $\{v_1, v_2, \ldots, v_l\}$ be a sequence of neighbors of s in the order such that $\forall_{1 \leq i < l} |N_T(v_i)| \geq |N_T(v_{i+1})|$. A vertex v_j $(1 \leq j \leq l)$ is **full** if and only if $|N_T(v_j) \setminus V(S)| = \max\{l - 2j - 2, 0\}$. The support vertex s is **saturated** if and only if $\forall_{1 \leq i \leq l} v_i$ is full.*

Definition 3 *By a **saturated star component** S in a tree T we mean a star component which has a support vertex $s \in V(S)$ that is saturated.*

Definition 4 *Let $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ be the following operations on a tree T.*

- *Operation $\mathcal{T}_1$: Attach a star S_{2l} $(l = 1 \vee l \geq 3)$ to a vertex v in a star component S of T, such that v is a leaf in S, v is not full and S is not saturated.*
- *Operation $\mathcal{T}_2$: Insert a path P_4 between an leaf-pendant star S of T and the subgraph $G[V(T) \setminus V(S)]$.*
- *Operation $\mathcal{T}_3$: Attach any number of star components S_{2l} $(l = 1 \vee l \geq 3)$ to pendant vertices of the last inserted P_4 subgraph.*

Let $\mathcal{F}_{ea}$ be the family of trees defined as $\mathcal{F}_{ea} = \{T : T$ is obtained from a star S_{2k} $(k = 1 \vee k \geq 3)$ by a finite sequence of operations $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3\} \cup \{S_4\}$.

Theorem 7 *Let T be a tree with $n(T) \geq 3$. Then, $\gamma_{ea}(T) = \frac{n(T)+s(T)}{2}$ if and only if $T \in \mathcal{F}_{ea}$.*

Definition 5 *Let $\mathcal{T}_4$, $\mathcal{T}_5$ and $\mathcal{T}_6$ be the following operations on a tree T.*

- *Operation $\mathcal{T}_4$: Attach a star S_{2l} $(l \geq 2)$ to a vertex v in a star component S of T, such that v is a leaf in S.*

- *Operation $\mathcal{T}_5$: Insert a path P_4 between an leaf-pendant star S of T and the subgraph $G[V(T) \setminus V(S)]$.*
- *Operation $\mathcal{T}_6$: Attach any number of star components S_{2l} ($l \geq 2$) to pendant vertices of the last inserted P_4 subgraph.*

Let $\mathcal{F}_{ca}$ be the family of trees defined as $\mathcal{F}_{ca} = \{T : T$ is obtained from a star S_{2k} ($k = 1 \vee k \geq 3$) by a finite sequence of operations $\mathcal{T}_4, \mathcal{T}_5$ and $\mathcal{T}_6\} \cup \{S_2\}$.

Theorem 8 *Let T be a tree with $n(T) \geq 3$. Then, $\gamma_{ca}(T) = \frac{n(T)+s(T)}{2}$ if and only if $T \in \mathcal{F}_{ca}$.*

3 Algorithms and complexity results

In the papers [7] and [8] the authors presented exact polynomial time algorithms for trees.

Theorem 9 *[7] There exists $O(n\Delta^2 \log \Delta)$ time algorithm finding the minimum global edge alliance for trees with at most n vertices and the maximum degree bounded by Δ.*

Theorem 10 *[8] There exists $O(n\Delta \log \Delta)$ time algorithm finding the minimum global complete alliance for trees with at most n vertices and the maximum degree bounded by Δ.*

In the paper [7] the authors proved the following NP-completeness result.

Theorem 11 *[7] The global edge alliance problem for subcubic graphs is NP-complete.*

In the paper [8] the authors proved

Theorem 12 *[8] The global complete alliance problem for subcubic bipartite graphs without pendant vertices is NP-complete.*

In the class of connected subcubic bipartite graphs without pendant vertices both problems global complete alliance problem and global edge alliance problem are equivalent. Thus,

Theorem 13 *[8] The global edge alliance problem for subcubic bipartite graphs without pendant vertices is NP-complete.*

4 Open problems

We conjecture that the global complete alliance number in trees may be bounded analogously to the global alliance number, i.e., let T be a tree with $n(T) \geq 3$, then $\gamma_{ca}(T) \leq \frac{3n(T)}{5}$.

In general graphs, we state two open problems. Let G be an arbitrary graph.

- Prove $\gamma_{ea}(G) \leq \frac{2n(G)}{3}$ or construct the counterexample.
- Prove $\gamma_{ca}(G) \leq \frac{3n(G)}{5}$ or construct the counterexample.

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