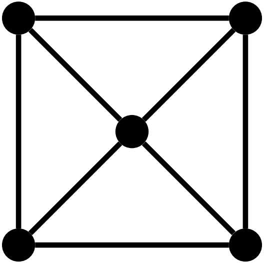
Johanna Wiehe:

The Chromatic Polynomial of a Digraph

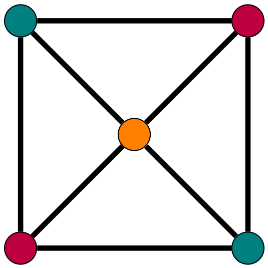
(joint work with W. Hochstättler)

FernUniversität in Hagen

undirected graphs

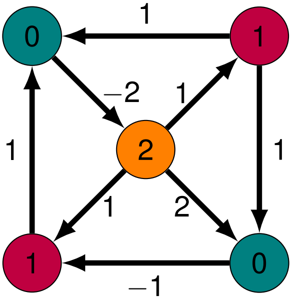


undirected graphs



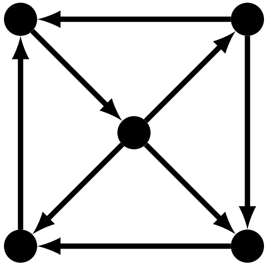
$\chi(G, k) = \#$ proper colorings with k colors

undirected graphs

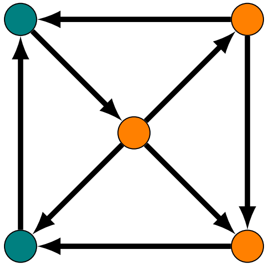


$\chi(G, k) = \#$ proper colorings with k colors
$= \#$ nowhere-zero k-coflows (i.e. k-tensions) in $G \cdot \#$ colors

directed graphs

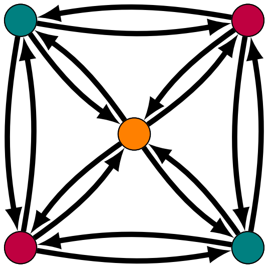


directed graphs



$\overrightarrow{\chi}(D, k) = \#$ acyclic colorings with k colors

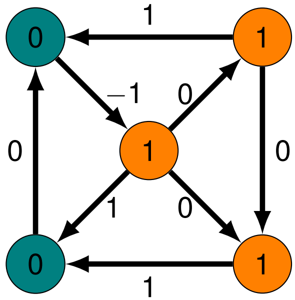
directed graphs: symmetric case



$\vec{\chi}(\overleftrightarrow{D}, k) = \#$ acyclic colorings with k colors

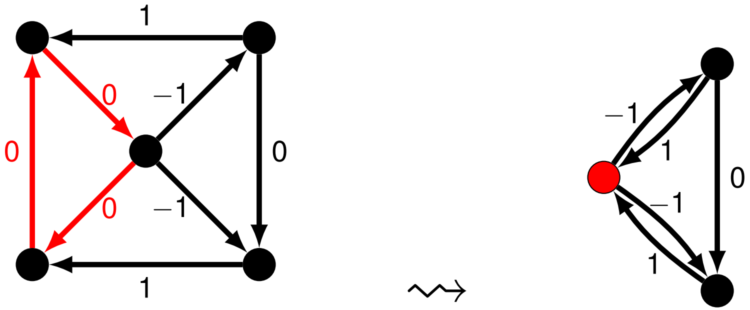
$= \chi(G[\overleftrightarrow{D}], k) = \#$ proper colorings with k colors

directed graphs



$\overrightarrow{\chi}(D, k) = \#$ acyclic colorings with k colors
$= \#$ Neumann-Lara k-coflows $\cdot \#$ colors
$= \psi_{NL}(D, k) \cdot \#$ colors

counting NL-coflows



the NL-coflow polynomial

Let $D = (V, A)$ be a digraph. Defining the poset $(\mathcal{C}, \supseteq)$ by

$$\mathcal{C} := \{A/C \mid \exists\, C_1, ..., C_r \text{ directed cycles, such that } C = \bigcup_{i=1}^{r} C_i\}$$

and using Möbius inversion, we find

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and using Möbius inversion, we find

$$\psi_{NL}(D, k) = \sum_{B \in \mathcal{C}} \mu(A, B) \cdot k^{rk(B)},$$

where μ is the Möbius function and $rk(B) = |V(B)| - c(B)$.

another representation

Consider the poset

$$\mathcal{P} = \{B \subseteq A \mid D[B] \text{ is totally cyclic subdigraph of } D\},$$

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$\Leftrightarrow x = (g(b_1), ..., g(b_m))^\top$ is solution of (1)

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where each element can be represented by a face of the polyhedral cone PC:

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$$\Rightarrow \psi_{NL}(D,k) = \sum_{B \in \mathcal{P}} (-1)^{rk_{\mathcal{P}}(B)} k^{rk(A/B)}$$

the chromatic polynomial of a digraph

$$
\begin{aligned}
\Rightarrow \overrightarrow{\chi}(D,k) &= \psi_{NL}(D,k) \cdot k^{c(D)} \\
&= \sum_{B \in \mathcal{P}} (-1)^{rk_{\mathcal{P}}(B)} k^{rk(A/B)+c(D)}
\end{aligned}
$$

symmetric digraphs

Let $D = (V, A)$ be a symmetric digraph and $G = (V, E)$ its underlying undirected graph. Then we have

$$\psi_{NL}(D, x) = \chi(G, x) \cdot x^{-c(G)}.$$

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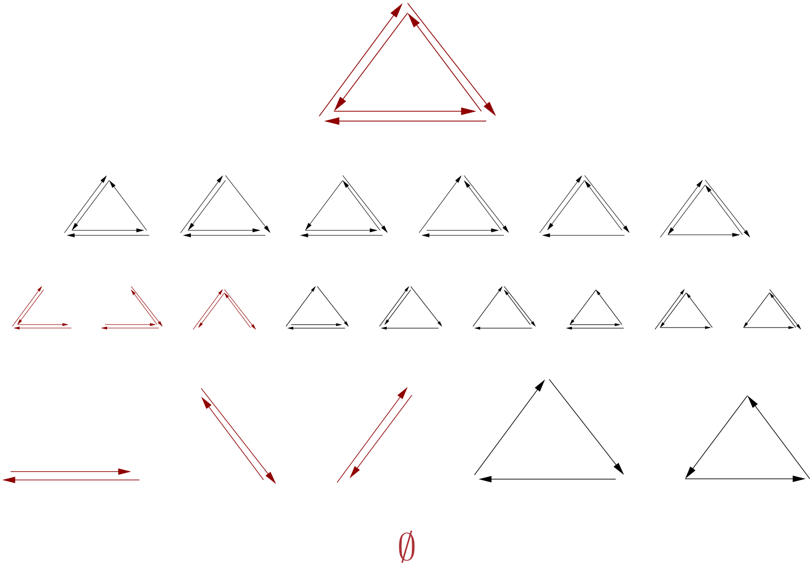
$$\psi_{NL}(D, x) = \chi(G, x) \cdot x^{-c(G)}.$$

$$\left(\Leftrightarrow \sum_{B \in \mathcal{P}} (-1)^{rk_{\mathcal{P}}(B)} k^{rk(A/B)+c(D)} = \sum_{B \subseteq E} (-1)^{|B|} k^{\tilde{c}(B)} \right)$$



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Sketch of proof. The polyhedra described by

$$\left.\begin{array}{rll} (M,-M)(\overrightarrow{x},\overleftarrow{x})^\top & = 0 & \\ \overrightarrow{x_i} + \overleftarrow{x_i} & \geq 1 & \forall 1 \leq i \leq m \\ \overrightarrow{x_i} & = 0 & \text{if } \overrightarrow{i} \notin A \text{ but } \overleftarrow{i} \in A \\ \overleftarrow{x_i} & = 0 & \text{if } \overleftarrow{i} \notin A \text{ but } \overrightarrow{i} \in A \\ \overrightarrow{x}, \overleftarrow{x} & \geq 0. & \end{array}\right\} (P)$$

are unbounded, thus their order complexes are contractible!

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are unbounded, thus their order complexes are contractible!

$$\Rightarrow \chi(P) - 1 = \mu(\emptyset, B) = 0$$

back to the general case

Theorem: Let $D = (V, A)$ be a digraph and $G = (V, E)$ its underlying undirected graph. Then

$$\psi_{NL}(D, x) = \sum_{B \in \mathcal{TC}} (-1)^{|B|} x^{\tilde{c}(B) - c(D)},$$

where $\mathcal{TC}$ includes all $B \subseteq E$ which admit a totally cyclic partial orientation $\mathcal{O}(B)$ in A such that $\bar{\mathcal{O}}(B)$ has no digons but bridges or $\bar{\mathcal{O}}(B)$ has a digon that is redundant but not a bridge.

open problems

- Conjecture(Neumann-Lara): *Every orientation of a simple planar graph has dichromatic number at most* 2.
- How does the NL-coflow polynomial of totally cyclic digraphs look like?
- In general, how does the NL-coflow polynomial of complete digraphs look like?
- Is there a meaningful two variable polynomial combining the dichromatic and the NL-flow polynomial as the Tutte polynomial does in the classical case?
- How many vertices suffice to create a 5-chromatic tournament? (a 3-chromatic tournament has at least 7 vertices, a 4-chromatic tournament at least 11)

Thank you for your attention.