Tight bounds on global edge and complete alliances in trees

Kacper Wereszko

Department of Algorithms and System Modeling
Faculty of Electronics, Telecommunications and Informatics
Gdańsk University of Technology

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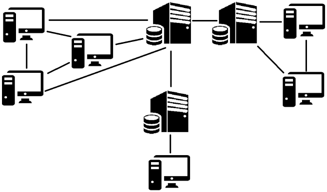
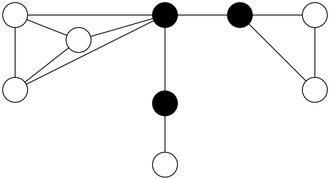
Joint work with **R. Kozakiewicz**, **R. Lewoń** and **M. Małafiejski**

Security problems in graphs

- Example: Sharing resources in computer network.
- We are looking for a configuration such that:
  - every user has access to at least one server,
  - none of the servers is overloaded,
  - the number of servers is as small as possible.





Neighborhood and domination

Let $G = (V, E)$ be a graph.

Neighborhood

For a set of vertices $S \subseteq V$, $N_G(S) = \{v \in V : \exists_{u \in S}\{u, v\} \in E\}$ is an **open neighborhood** of S, and $N_G[S] = N_G(S) \cup S$ is a **closed neighborhood** of S.

Dominating set

A set $D \subseteq V$ is a **dominating set** of G iff every vertex from $V \setminus D$ has at least one neighbor in the set D. A set $D_t \subseteq V$ is a **total dominating set** of G iff every vertex from V has at least one neighbor in the set D_t.

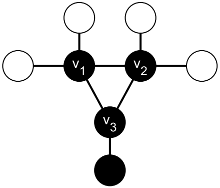
Note that $N_G[D] = V$ and $N_G(D_t) = V$

Predicate *SEC*

Security predicate

Let $G = (V, E)$. The **security predicate (SEC)** for a set of vertices $S \subseteq V$ ("set of defenders") and a subset $X \subseteq S$ is defined as follows:

$$SEC_{G,S}(X) = true \iff |N_G[X] \cap S| \geq |N_G[X] \setminus S|$$



- $\forall_{v \in \{v_1, v_2, v_3\}} SEC_{G,S}(\{v\}) = true$
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Global defensive alliance (GDA)

Let $G = (V, E)$.

Defensive alliance

Set $S \subseteq V$ is a **defensive alliance**, if and only if

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Global defensive alliances in graphs (2003)
Haynes, Hedetniemi, Henning

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- Defensive alliance is **global** if it is also a dominating set of G.
- Defensive alliance is **total** if it is also a total dominating set of G.

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- $\gamma_a(G)$ – the size of the minimum GDA in G (global defensive alliance number of G).

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Global edge alliance (GEA)

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Set $S \subseteq V$ is an **edge alliance**, if and only if $G[S]$ has no isolated vertices, and

$$\forall_{\{u,v\} \in E(G[S])} SEC_{G,S}(\{u, v\}) = true$$

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Lewoń, Małafiejska, Małafiejski, W

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- $\gamma_{ea}(G)$ – the size of the minimum GEA in G (global edge alliance number of G).

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Global complete alliance (GCA)

Let $G = (V, E)$.

Complete alliance

Set $S \subseteq V$ is a **complete alliance**, if and only if for each clique (i.e. complete subgraph) K in $G[S]$ we have

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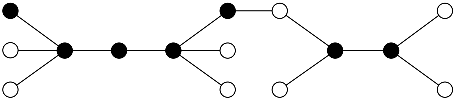
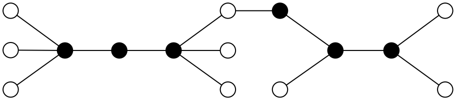
Comparison



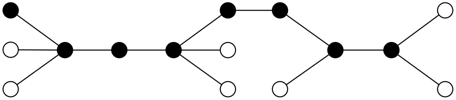
Global **defensive** alliance

$\gamma_a(G) = 7$



Global **edge** alliance

$\gamma_{ea}(G) = 6$



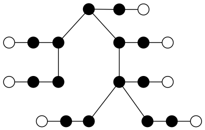
Global **complete** alliance

$\gamma_{ca}(G) = 8$

Total domination number - upper bounds

Some known upper bounds on total domination number:

- $\gamma_t(G) \leq \frac{2n(G)}{3}$ [Cockayne, Dawes, Hedetniemi (1980)]
  - for general graphs
  - $n(G) \geq 3$
  - tight
- $\gamma_t(T) \leq \frac{n(T)+s(T)}{2}$ [Haynes, Chellali (2004)]
  - only for trees
  - $n(T) \geq 3$
  - tight

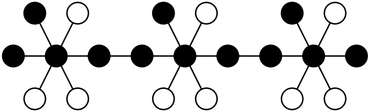


Total dominating set in 2-corona of a tree.

Global defensive alliance number - upper bounds

Some known upper bounds on global defensive alliance number:

- $\gamma_a(T) \leq \frac{n(T)+s(T)}{2}$ [Chen, Shiu (2011)]
  - only for trees
  - $n(T) \geq 3$
  - tight

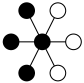


A tree T for which $\gamma_a(T) = \frac{n(T)+s(T)}{2}$.

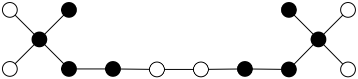
Global complete alliance upper bound in trees: $\frac{n+s}{2}$

For every tree T of order $n(T) \geq 3$, $\gamma_{ca}(T) \leq \frac{n(T)+s(T)}{2}$

- tight for the same family of trees as $\gamma_a(T)$.



A star S_6
$\forall_{k \in \mathbb{N}} \gamma_{ca}(S_{2k}) = \frac{n+s}{2}$



A tree T for which $\gamma_{ca}(T) = \frac{n(T)+s(T)}{2}$.

Characterization of trees with $\gamma_{ca}(T) = \frac{n+s}{2}$

Let $\mathcal{F}_{ca}$ be a family of trees T, where:

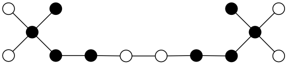
- T is a star of odd order, OR
- T is obtained from a number $s \geq 2$ of odd order star graphs (with the exception of S_2), and any number t of P_4 graphs, by adding $s + t - 1$ edges between leaves of these graphs in such a way that:
  - the center of each star S_{2k} is adjacent to at least $1 + k$ leaves in T, AND
  - each leaf of every P_4 graph is incident to at least one added edge.

1. For every tree T of $n(T) \geq 3$ $\gamma_a(T) = \frac{n+s}{2}$ iff $T \in \mathcal{F}_{ca}$[1]
2. For every tree T of $n(T) \geq 3$ $\gamma_{ca}(T) = \frac{n+s}{2}$ iff $T \in \mathcal{F}_{ca}$

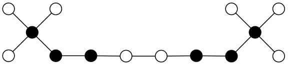
[1] *A new upper bound on the global defensive alliance number in trees* (2011) Chen, Shiu

Global edge alliance - the same upper bound?

- For some trees $T \in \mathcal{F}_{ca}$ we have $\gamma_{ea}(T) < \gamma_a(T) = \gamma_{ca}(T) = \frac{n(T)+s(T)}{2}$



$$\gamma_{ca}(T) = \gamma_a(T) = \frac{n(T)+s(T)}{2} = 8$$

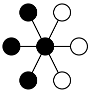


$$\gamma_{ea}(T) = 6 < \frac{n(T)+s(T)}{2}$$

Global edge alliance upper bound in trees: $\frac{n+s}{2}$

For every tree T of order $n(T) \geq 3$,
$\gamma_{ea}(T) \leq \frac{n(T)+s(T)}{2}$

- tight
  - some examples are shown below



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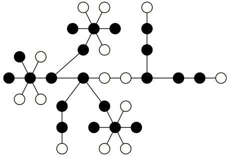


A tree T for which $\gamma_{ea}(T) = \frac{n+s}{2}$

Star components

Definition

A **star component** S of a tree T is a subgraph of T induced by a set of vertices $N[s]$, where $s \in V(T)$ is a support vertex in T.

Definition

A **pendant star** S in a tree T is a star component of T such that exactly one edge $e \in E(T)$ connects S with the subgraph $G[V(T) \setminus V(S)]$. If $v \in e, v \in V(T) \setminus V(S)$ is not a support vertex in T, then S is an **l-pendant star**.

Full and saturated vertices

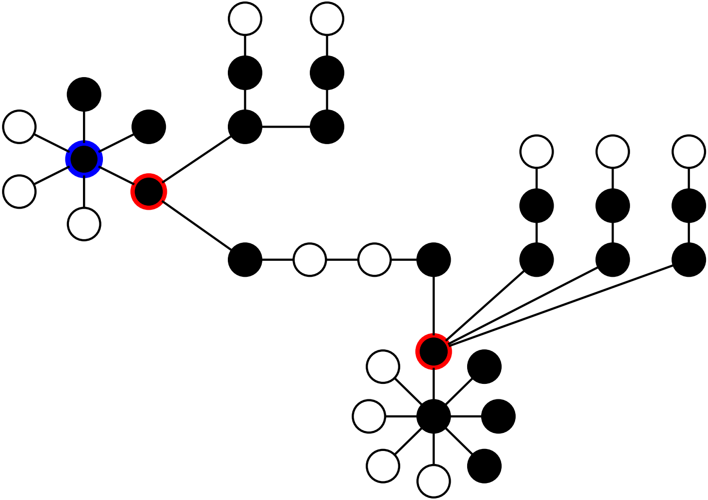
Definition

Let S be a star component of a tree T and let $v_s \in V(S)$ be a support vertex in S. Let $l = |N_S(v_s)|$. Let $\{v_1, v_2, \ldots, v_l\}$ be a sequence of neighbors of v_s in the order such that $\forall_{1 \leq i < l} |N_T(v_i)| \geq |N_T(v_{i+1})|$. A vertex v_j $(1 \leq j \leq l)$ is **full** if and only if $|N_T(v_j) \setminus V(S)| = \max\{l - 2j - 2, 0\}$. The support vertex s is **saturated** if and only if $\forall_{1 \leq i \leq l} v_i$ is full.

$S_2 = P_3$ - special case!

- one of the leaves v_1 is never full ($deg(v_1)$ can be arbitrarily large)
- the other one, v_2, is always full ($deg(v_2) = 1$)

Example



Characterization of trees with $\gamma_{ea}(T) = \frac{n+s}{2}$

Let $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ be the following operations on a tree T.

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Let $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$ be the following operations on a tree T.

- **Operation** $\mathcal{T}_1$: Attach a star S_{2l} ($l = 1 \vee l \geq 3$) to a vertex v in a star component S of T, such that v is a leaf in S, v is not full and S is not saturated.

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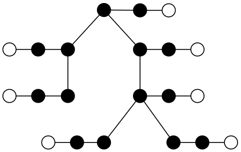
Let $\mathcal{F}_{ea}$ be the family of trees defined as $\mathcal{F}_{ea} = \{T : T$ is obtained from a star S_{2k} ($k \geq 1$) by a finite sequence of operations $\mathcal{T}_1, \mathcal{T}_2$ and $\mathcal{T}_3\}$.

Global edge alliance upper bound in trees: $\frac{2n}{3}$

Let T be a tree of order $n(T) \geq 3$.
Then, $\gamma_{ea}(T) \leq \frac{2n(T)}{3}$

- tight (example: 2-corona of a tree)



Global edge alliance in 2-corona of a tree.

Open problems

Question

Is it true that $\gamma_{ca}(G) \leq \frac{3n}{5}$ if G is an arbitrary graph with $n(G) > 1$?

Question

Is it true that $\gamma_{ea}(G) \leq \frac{2n}{3}$ if G is an arbitrary graph with $n(G) > 2$?

Thank you for your attention!