The unsuitable neighbourhood inequalities for the fixed cardinality stable set polytope

18th Cologne-Twente Workshop on Graphs and Combinatorial Optimization

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September 15th, 2020

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From conflict-free spanning trees to fixed cardinality stable sets

Minimum spanning trees under conflict constraints (MSTCC)

Input

- simple undirected graph $G(V, E)$
- set $C \subset E \times E$ of conflicting edge pairs
- edge weights $w : E \to \mathbb{Q}_+$

Output: if feasible, a subset $T \subseteq E$ satisfying

- (V, T) is a spanning tree of G
- at most one of edges e_i and e_j is in T, for each pair $(e_i, e_j) \in C$
- T has the minimum weight $\sum_{e \in T} w(e)$ out of all *conflict-free* spanning trees

MSTCC: definition using the conflict graph $\hat{G}(E, C)$

Feasible solution for the MSTCC problem
Subset of E corresponding simultaneously to a spanning tree of $G(V, E)$ and a *stable set* of $\hat{G}(E, C)$.

$$\min \left\{ \sum_{e \in E} w(e) x_e : \mathbf{x} \in P_{sptree(G)} \cap P_{stab(\hat{G})} \cap \{0, 1\}^{|E|} \right\}$$

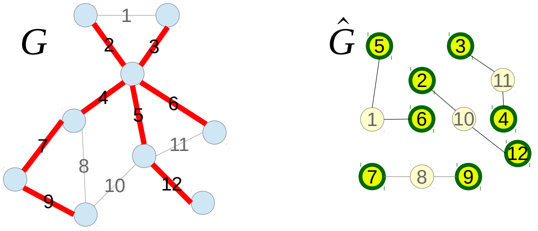

Figure 1: The original graph G and the conflict graph $\hat{G}$, with a feasible solution highlighted.

Fixed cardinality stable sets

Input

- simple undirected graph $G(V, E)$
- $k \in \mathbb{Z}_+$
- vertex weights $w : V \to \mathbb{Q}_+$

Output: if feasible, a subset $T \subseteq V$ satisfying

- T is a stable set of G
- $|T| = k$
- T has the minimum weight $\sum_{u \in T} w(u)$ out of all k-stabs in G

Fixed cardinality stable sets: a gap in the literature

[Bruglieri et al., 2006]
An annotated bibliography of combinatorial optimization problems with fixed cardinality constraints

Brief appearances

- [Janssen and Kilakos, 1999] for $k \in \{2, 3\}$
- [Botton, 2010] algorithm for a variant of the survivable network design problem
- Parameterized extension complexity: [Bazzi et al., 2019], [Gajarsk et al., 2018], [Buchanan and Butenko, 2014], [Buchanan, 2016].

[Mannino et al., 2007]
The stable set problem and the thinness of a graph

Polyhedral results

Our objects

$$\mathfrak{C}(G,k) = \textbf{conv}\left\{\chi^S \in \{0,1\}^V : S \subset V \text{ induces a stable set}, |S| = k\right\}$$

$\mathcal{P}(G,k)$ denotes the polyhedral region defined by

$$\sum_{v \in V} x_v = k \qquad (1)$$

$$x_u + x_v \leq 1 \qquad \forall \{u,v\} \in E \qquad (2)$$

$$0 \leq x_v \leq 1 \qquad \forall v \in V \qquad (3)$$

$\mathcal{P}(G, k)$ is *no longer* half-integer

- Recall that a vector z is *half-integer* if $2z$ is integer. (More generally, z is $\frac{1}{p}$-*integer* if pz is integer.)
- [Nemhauser and Trotter, 1974]: the *fractional stable set polytope* is half-integer, *i.e.* all its vertices are $\{0, \frac{1}{2}, 1\}$-valued.

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For each $p \geq 2$ and each $k \geq 2$, there exists a graph G such that $\mathcal{P}(G, k)$ is not $\frac{1}{p}$-integer.

Wanted: valid inequalities for $\mathfrak{C}(G, k)$

Neighbourhood of $S \subset V$:

$$N(S) = \{u \in V \setminus S : \exists \{u, v\} \in E \text{ for some } v \in S\}$$

Neighbourhood of a vertex $v \in V$:

$$\delta(v) = N(\{v\})$$

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Proposition
If $\mathbf{x}$ *is the incidence vector of any k-stab, and* $v \in V$ *is such that* $|\delta(v)| > n - k$*, then* $x_v = 0$*.*

Unsuitable neighbourhood inequalities (UNI)

Theorem
For each $S \subset V$ such that $1 \leq |S| < k$ and $|N(S)| > n - k$, inequality $\sum_{v \in S} x_v \leq |S| - 1$ is valid for $\mathfrak{C}(G, k)$.

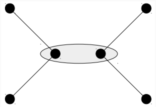

Figure 2: $G = 2P_3$ and $k = 3$

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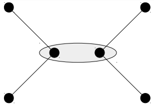

Figure 2: $G = 2P_3$ and $k = 3$. Then, $|\delta(u)| \leq n - k = 3$ for each vertex u.

Theorem
For any graph G and $k > 1$, the UNI imply the condition enforced by the previous proposition in the description of $\mathfrak{C}(G, k)$, but the converse does not hold.

Unsuitable neighbourhood inequalities (UNI)

Proposition
In either of the following two conditions, the corresponding unsuitable neighbourhood inequality is redundant in $\mathfrak{C}(G, k)$:

(i) *if $S \subset V$ is not independent, or*

(ii) *if $S \subset V$ is not minimal with respect to the condition $|N(S)| > n - k$.*

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UNI and rank inequalities from the classical stable set polytope

$$\sum_{v \in W} x_v \leq \alpha(G[W]), \text{ for } W \subset V(G)$$

- $\alpha(G[W]) = |W|$ whenever W is an independent set
- UNI over W: $\sum_{v \in W} x_v \leq |W| - 1$

Towards a branch-and-cut algorithm

Separation problem for UNI

Given a graph $G = (V, E)$, with $n = |V|$, $k \in \{2, \ldots, n-1\}$, and $x^* \in [0, 1]^n$ satisfying the conditions that $\sum_{v \in V} x_v^* = k$ and that $x_u^* + x_v^* \leq 1$ for each $\{u, v\} \in E$, determine

i. either a set $S \subset V$, with $1 \leq |S| \leq k-1$ and $|N(S)| \geq n-(k-1)$, such that $\sum_{v \in S} x_v^* > |S| - 1$, in which case the unsuitable neighbourhood inequality corresponding to S separates x^* from $\mathfrak{C}(G, k)$,

ii. or that no such set exists, in which case all UNI are satisfied at x^*.

Separation problem for UNI, equivalent formulation

Given the input $[G, k, x^*]$ corresponding to the previous definition, define $y^* \in [0, 1]^n$ such that $y^*_v = 1 - x^*_v$.

Then, $\sum_{v \in S} x^*_v > |S| - 1$ if and only if $\sum_{v \in S} y^*_v < 1$.

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Given a graph $G = (V, E)$, with $n = |V|$, $k \in \{2, \ldots, n-1\}$, and $y^* \in [0,1]^n$ satisfying the conditions that $\sum_{v \in V} y_v^* = n - k$ and that $y_u^* + y_v^* \geq 1$ for each $\{u, v\} \in E$, determine

1. either a set $S \subset V$, with $|N(S)| \geq n - (k-1)$ and $\sum_{v \in S} y_v^* < 1$, in which case the unsuitable neighbourhood inequality corresponding to S separates $x^* = \mathbf{1} - y^*$ from $\mathfrak{C}(G, k)$,
2. or that no such set exists, in which case all UNI are satisfied at $x^* = \mathbf{1} - y^*$.

Separation problem for UNI, equivalent formulation

If $|S| = k - 1$**, then** $|N(S)| \geq n - (k - 1)$ **implies that it would be a dominating set**

- Recall that adjacent vertices have y^* values summing up to at least 1
- Since we require $\sum_{v \in S} y_v^* < 1$, we would actually have an *independent dominating set* if $|S| = k - 1$.

Allowing $|S| \leq k - 1$
means that there might be $q \in \{0, 1, \ldots, k - 2\}$ vertices neither in S nor dominated by it.

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Definition
A q-quasi dominating set in a graph $G = (V, E)$ is a subset of vertices which is dominating in $G[V \backslash X]$, for some $X \subset V$, $|X| \leq q$.

Separation problem for UNI
Find a $(k-2)$-quasi dominating set of weight at most 1, or decide that none exists.

The balanced branching rule of [Balas and Yu, 1986]

On a given node of the enumeration tree

- $G' = (V', E')$ denotes the subgraph induced by vertices not fixed in this subproblem
- $\bar{z}$ denotes the best primal bound available.

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An "easy piece" $W \subseteq V'$

- Let $W \subseteq V'$ be such that we can determine efficiently that the minimum weight of a k-stab in the subgraph induced by W, denoted $z(W)$, is such that $z(W) \geq \bar{z}$.

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Partition the search space into the sets

$$V'_i = \{v_i\} \bigcup V' \backslash \left(N(v_i) \cup \{v_{i+1}, \ldots, v_p\}\right), \text{ for } 1 \leq i \leq p.$$

A combinatorial dual bound

Theorem
Let $M \subset E$ be any matching in G. Define:

- $c_e = \min \{w(v_i), w(v_j)\}$ *for each edge* $e = \{v_i, v_j\} \in M$
- $c_u = w(v_u)$ *for any vertex* v_u *not covered by the matching* M

Then, the sum of the k *lowest values in the image of* $c(\cdot)$ *is a lower bound on* $z = \min \left\{ \sum_{v \in V} w(v) x_v : \boldsymbol{x} \in \mathcal{P}(G, k) \cap \{0, 1\}^n \right\}$*.*

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Determine candidate subgraphs W by inspecting, for each $l \in \{1, \ldots, k\}$

1. A minimum $c(\cdot)$-weighted matching in G' with cardinality l
2. A suitable choice of $k - l$ vertices not covered by the matching

Concluding remarks

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Polyhedral investigation
Many open questions about $\mathfrak{C}(G,k)$.

Already for $n = 4$, the nine non-isomorphic graphs on 4 vertices (discarding the empty and the complete graphs) give $\dim \mathfrak{C}(G,2) \in \{0,1,2,3\}$ and $\dim \mathfrak{C}(G,3) \in \{-1,0,1\}$.

The UNI separation problem and its complexity
Optimizing over subgraphs with a domination-like property and an additional budget constraint

Balanced branching scheme
Leveraging a modern branch-and-cut solver for the classical stable set problem towards one for the fixed-cardinality version

Thank you!

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