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Piecewise Linear Valued Constraint Satisfaction Problems with Fixed Number of Variables

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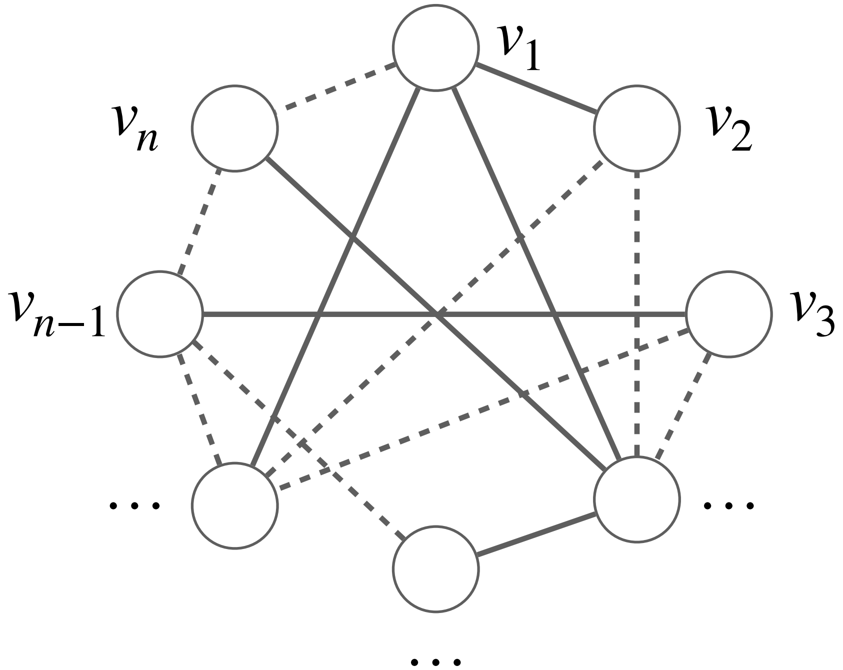
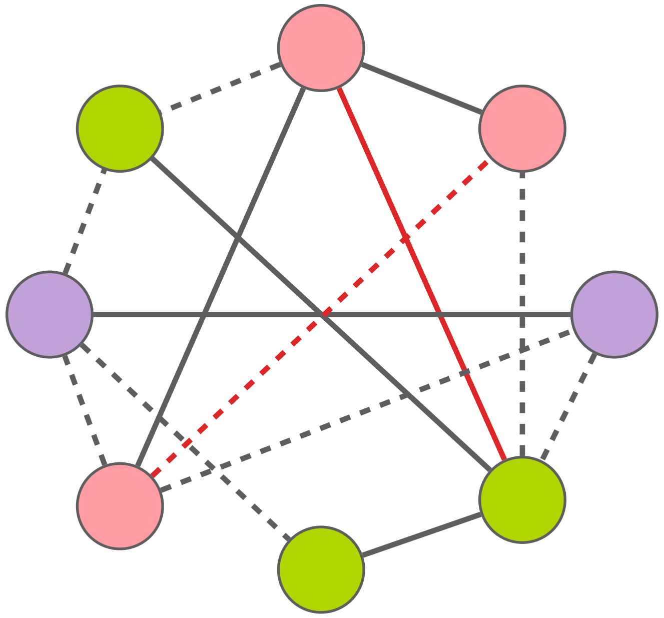
Minimun Correlation Clustering with Partial Information

- A finite set of invited people $V := \{v_1, \ldots, v_n\}$;
- an arbitrary number of tables $T := \{t_1, t_2, t_3, \ldots\}$;
- E_1: a list of couples of guests that want to sit at the same table (solid edges);
- E_2: a list of couples of guests that want to sit at different tables (dashed edges).

Task: find an assignment of tables to the invited people that minimises the sum of the couples from E_1 that are assigned to different tables and the couples from E_2 that are assigned to the same table.





An assignment with cost 2.

Valued Constraint Satisfaction Problems

Valued structure: a set Γ of cost functions over a domain D.

An instance of the **valued constraint satisfaction problem** for a valued structure Γ, $VCSP(\Gamma)$, is given as an objective function, i.e., an expression $\phi(x_1, \ldots, x_n)$ of the form

$$\sum_{i=1}^{m} f_i(x_1^i, \ldots, x_{ar(f_i)}^i)$$

where $f_1, \ldots, f_m \in \Gamma$ and the x_j^i are variables from $V = \{x_1, \ldots, x_n\}$.

Task: minimise the objective function, that is, find

$$\inf \sum_{i=1}^{m} f_i(\alpha(x_1^i), \ldots, \alpha(x_{ar(f_i)}^i)).$$

for $\alpha : V \to D$.

VCSPs over infinite domains

Theorem (Kozik-Ochremiak, 2015 + Thapper-Živný, 2016 + Kolmogorov-Krokhin-Rolinek, 2017)

The VCSP for any **finite-domain valued structure** is either in P or NP-complete.

(Subject to the computational complexity dichotomy for finite-domain CSPs proved by Bulatov and by Zhuk in 2017.)

Examples of infinite-domain VCSPs:

- Minimum Correlation Clustering with Partial Information (NP-complete);
- Minimum Feedback Arc-Set (NP-complete);
- Linear Programming (Polynomial-time tractable);
- Linear Least Square (Polynomial-time tractable).

Theorem (Bodirsky-Grohe, 2008)

For every computational decision problem L there exists an infinite-domain valued structure Γ such that L is Turing-equivalent to VCSP(Γ).

Piecewise Linear Cost Functions

A cost function $f\colon \mathbb{Q}^n \to \mathbb{Q} \cup \{+\infty\}$ is piecewise linear (PL) if its domain, $dom(f)$, can be represented as the union of finitely many polyhedral sets, relative to each of which $f(x)$ is given by a linear expression.

We adopt the convention $f(x) = +\infty \Leftrightarrow x \notin dom(f)$.

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Can be encoded as VCSPs using PL cost functions

Proposition

Let Γ be the valued structure containing all PL cost functions. Then $VCSP(\Gamma)$ is NP-complete.*

* The computational complexity of the VCSP for a valued structure containing infinitely many cost functions **depends on the representation of the input**.

PL VCSPs with a fixed number of variables

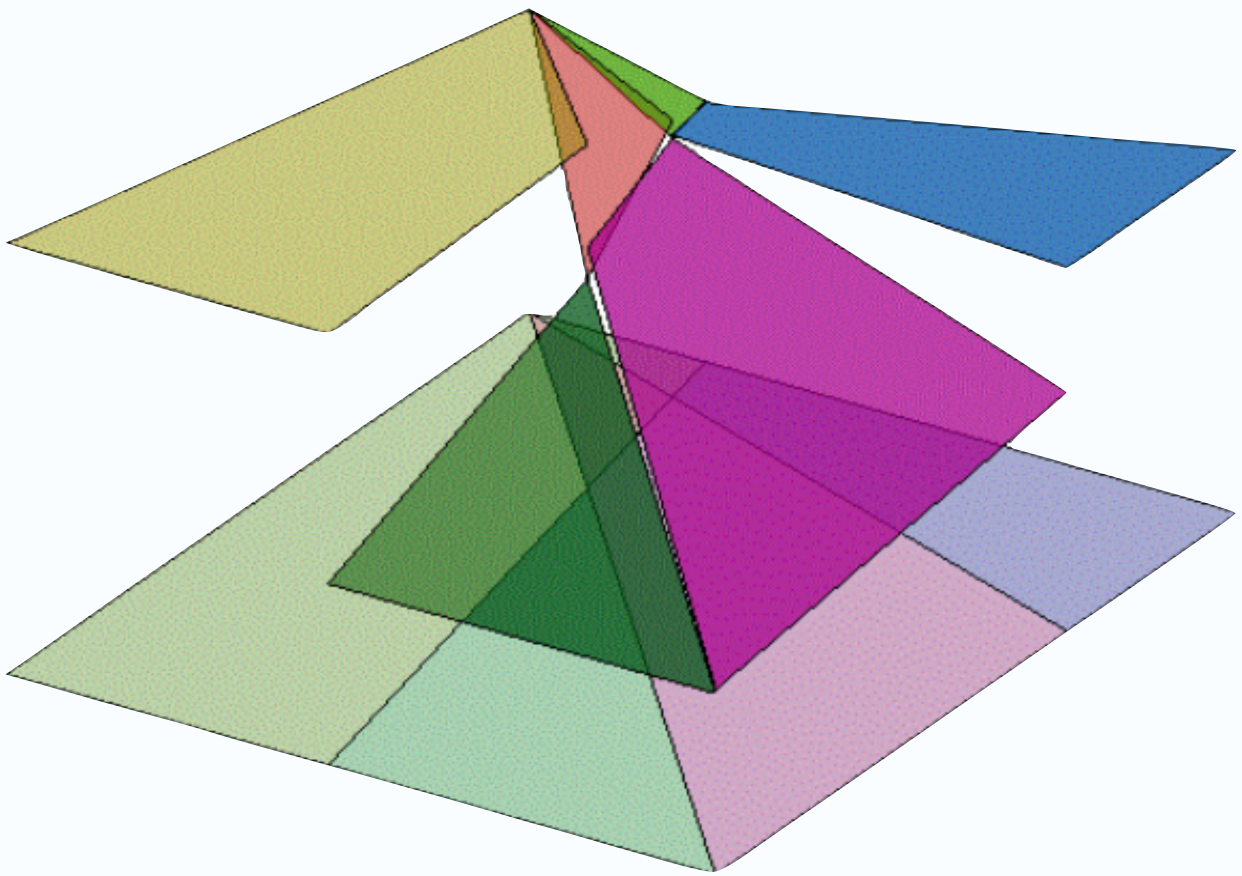
In the case of a valued structure Γ containing infinitely many cost functions, it makes sense to consider the restriction of $VCSP(\Gamma)$ to instances over a **fixed set of variables**.

Theorem (Bodirsky-Mamino-**V**., 2020)

Let V be a finite set of variables. Then there is a polynomial-time algorithm that solves the VCSP for all PL cost functions having variables in V. *

* The computational complexity of such an algorithm depends on the representation of the cost functions in the input.

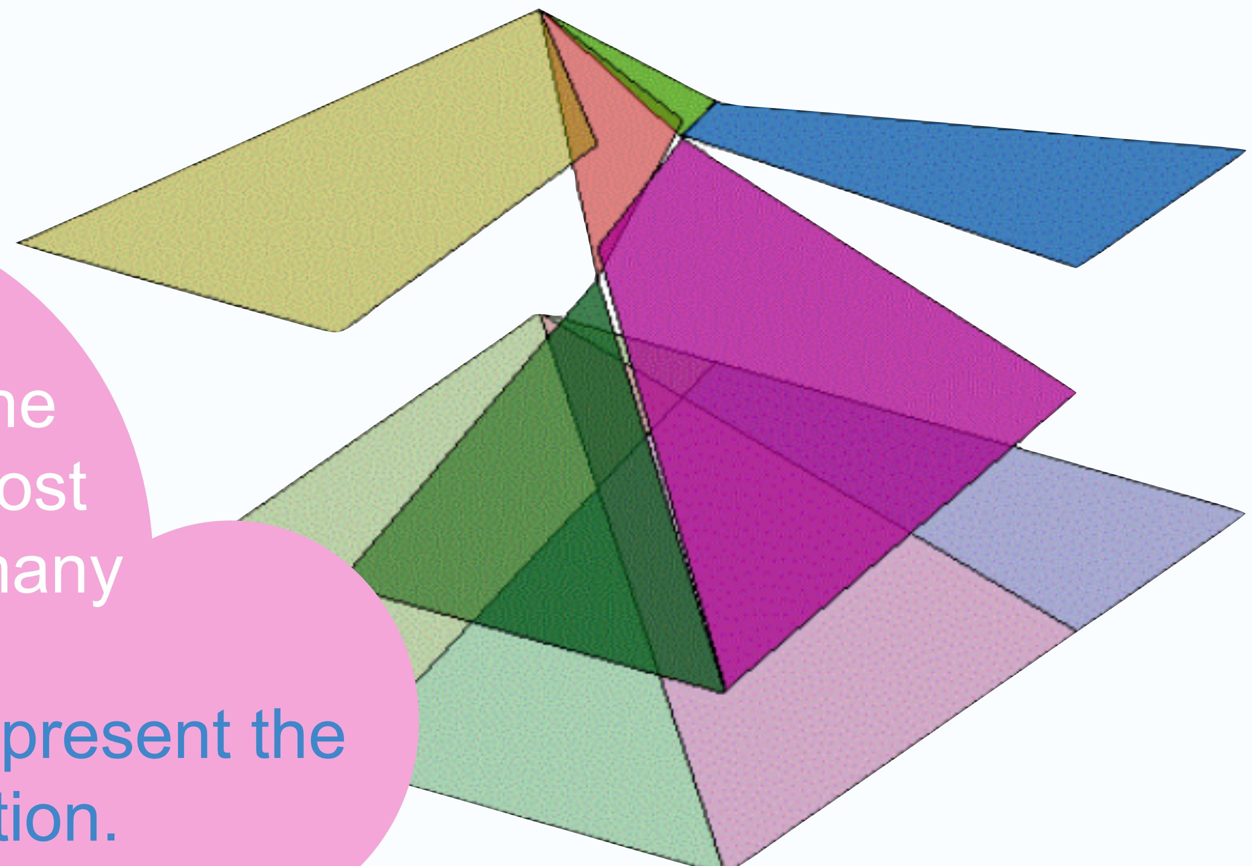
Representing PL Cost Functions



Representation of a PL cost function: **a list of linear constraints**, specifying the polyhedral sets in which the domain is partitioned, and **a list of linear polynomials**, defining the value of the function relatively to each polyhedral set.

- The linear constraints and the linear polynomials are encoded by the list of their rational coefficients.
- The constants for (numerators and denominators of) rational coefficients are represented in binary.

Representing PL Cost Functions



Size of an instance of the VCSP for PL cost functions (with fixed many variables):= # of bits required to represent the objective function.

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A Poly-Time Algorithm for PL VCSPs with n variables

Theorem (Bodirsky-Mamino-**V**., 2020)

Let V be a finite set of variables. Then there is a polynomial-time algorithm that solves the VCSP for <u>all</u> PL cost functions having variables in V.

Idea: compute the feasible regions of linearity and solve a linear program in each of them.

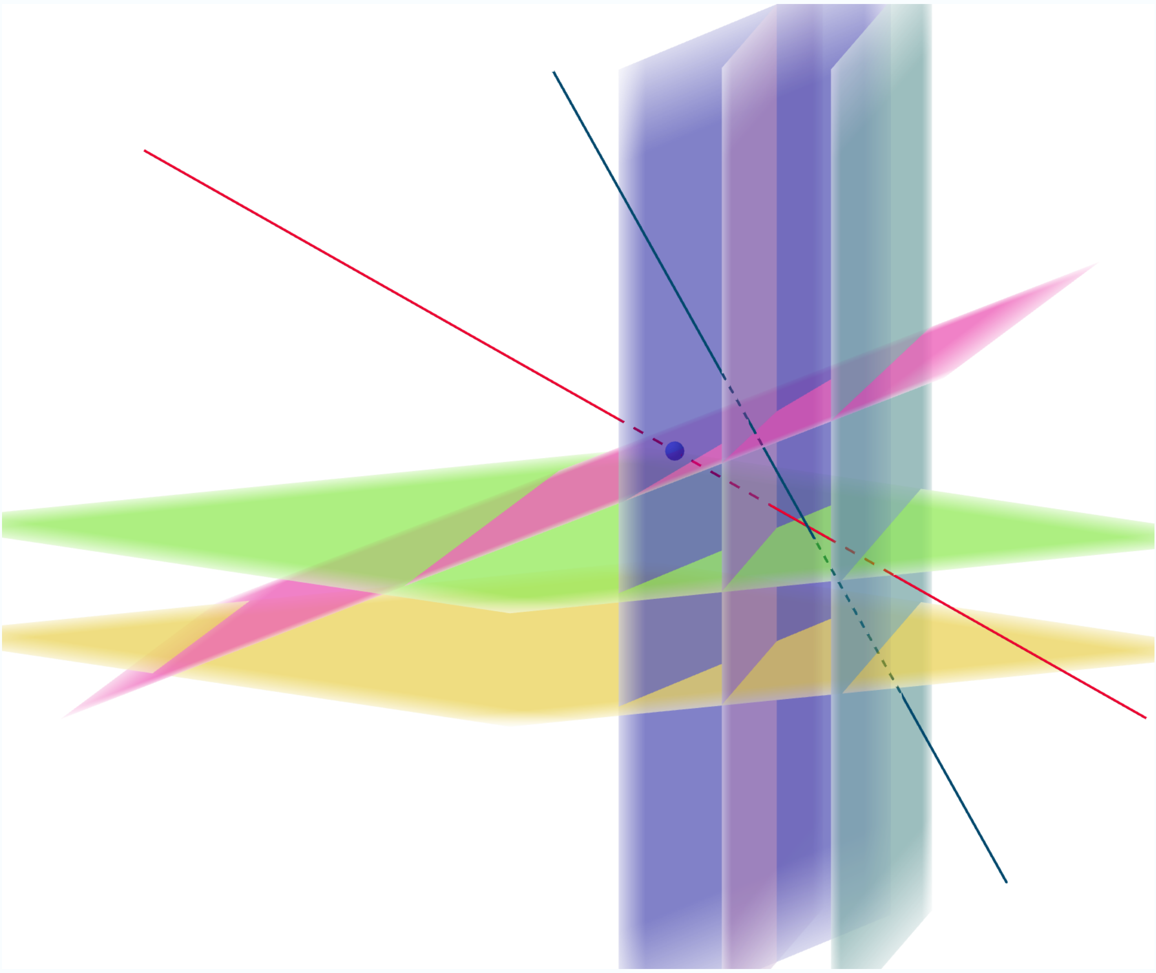
How many regions of linearity are there?

A priory, exponentially many in the size of the instance.

Actually, the regions of linearity are bounded by a polynomial in k

$$\sigma \leq \tau_d(k) = \sum_{i=0}^{d} 2^i \binom{k}{i}$$

where k := # of polynomials appearing in the finite set of linear constraints defining the domain of some cost function in the input.



Linear Programming with Strict Linear Inequalities

In a LP instance the linear constraints defining the convex domain of a linear cost function are usually linear weak inequalities ($a^\top x \le b$). Our algorithm can deal also with linear strict inequalities ($a^\top x < b$) by an application of Motzkin Transposition Theorem.

Theorem (Motzkin, 1936)

Let $A \in \mathbb{Q}^{k_1 \times d}$, $B \in \mathbb{Q}^{k_2 \times d}$ be matrices such that $\max(k_1, d) \ge 1$. The system

$$\begin{cases} Ax < 0 \\ Bx \le 0 \end{cases}$$

has a solution $x \in \mathbb{Q}^d$ if, and only, if the system

$$\begin{cases} A^T y + B^T z = 0 \\ y \ge 0, z \ge 0 \end{cases}$$

does not admit a solution $(y, z) \in \mathbb{Q}^{k_1+k_2}$ such that $y \ne (0,\ldots,0)$.

Let $A \in \mathbb{Q}^{k_1 \times d}$, $B \in \mathbb{Q}^{k_2 \times d}$. To solve the Linear Programming Feasibility with constraints $Ax < 0$, $Bx \le 0$, we solve the linear program

$$\min \sum_{j=1}^{k_1+1} (-y_j) \quad \text{subject to}$$

$$A^T y + B^T z = 0$$
$$-y \le 0$$
$$-z \le 0.$$

If this is infeasible or has an optimal solution with the first k_1 coordinates equal 0, then $Ax < 0$, $Bx \le 0$ has a solution and we accept; otherwise we reject.

What is next: semialgebraic VCSPs with fixed number of variables

A function $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is called semialgebraic if its domain can be represented as the union of finitely many basic semialgebraic sets of the form $\{x \in \mathbb{R}^n \mid \chi(x)\}$where χ is a conjunction of (weak or strict) polynomial inequalities with integer coefficients, relative to each of which $f(x)$ is given by a polynomial expression with integer coefficients.

In general, the VCSP for all semialgebraic cost function is equivalent to the existential theory of the reals, which is in PSPACE.

The restriction of the feasibility problem associated with a semialgebraic VCSP to a fixed number of variables is solvable in polynomial-time by cylindrical decomposition.

Thank you