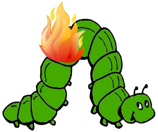
On the Burning Number of p-Caterpillars



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Lehrstuhl II für Mathematik
RWTH AACHEN UNIVERSITY

Overview

- **burning number** $b(G)$ of an undirected graph $G = (V, E)$:
  $\rightarrow$ minimum number of time steps to inflame the whole graph

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- ✓ Hamiltonian graphs
- ✓ Spiders
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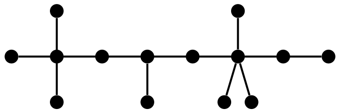
Trees!

Example



- Central Spine: $P_\ell = P_8$
- Maximum leg length: $p = 1 \rightarrow$ 1-caterpillar

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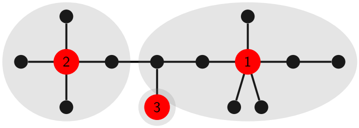

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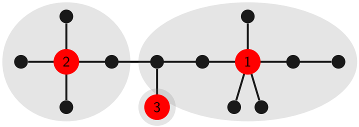

- Central Spine: $P_\ell = P_8$
- Maximum leg length: $p = 1 \to$ 1-caterpillar

Observation

For a 1-caterpillar G, we have $\lceil\sqrt{\ell}\rceil \leq b(G) \leq \lceil\sqrt{\ell}\rceil + 1$.

Theorem (Burning Number Conjecture for 1-Caterpillars)

The burning number of any 1-caterpillar G satisfies $b(G) \leq \lceil \sqrt{n} \rceil$.

Proof sketch for 1-Caterpillars

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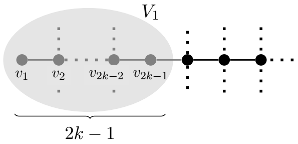

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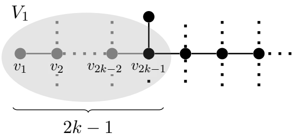

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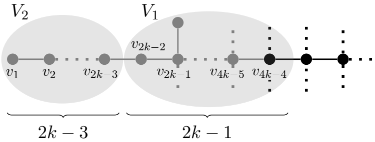

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Theorem ($\mathcal{NP}$-completeness)

The burning problem is $\mathcal{NP}$-complete for 1-caterpillars of maximum degree three.

Problem: DISTINCT 3-PARTITION

Instance: A set $X = \{a_1, \ldots, a_{3n}\}$ of $3n$ distinct positive integers and a positive integer S, fulfilling $\sum_{i=1}^{3n} a_i = nS$ with $\frac{S}{4} < a_i < \frac{S}{2}$ for all $1 \leq i \leq 3n$.

Question: Can the set X be partitioned into n triples each of whose elements sum up to S?

Complexity

- $m := \max\{a_i \mid a_i \in X\}$, $\underline{m} := \{1, \ldots, m\}$ and $Y := \underline{m} \setminus X$
- universe of Burning Number: $X' := \{2a_i - 1 \mid a_i \in X\}$, $S' := 2S - 3$,
  $\mathcal{O}_m := \{2i - 1 \mid i \in \underline{m}\}$, $Y' := \mathcal{O}_m \setminus X'$

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- for the n triples (each of whose elements should add up to S) $\rightarrow$ paths $Q_1, \ldots, Q_n$ of length S'
- for all elements in Y $\rightarrow$ separate path $R_{y'}$ of length y' for all $y' \in Y'$

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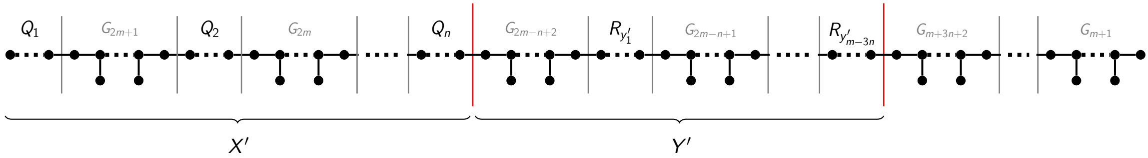
- connect paths by caterpillars to keep them separated:
  $\rightarrow G_{2m+1}, \ldots, G_{m+1}$, where G_i has spine length $2i - 1$ for $2m + 1 \geq i \geq m + 1$

Complexity



- longest path P_ℓ with

$$\ell = \left|\bigcup_{i=1}^{n} V\left(Q_i\right)\right| \cup \left|\bigcup_{y' \in Y'} V\left(R_{y'}\right)\right| \cup \left|\bigcup_{i=m+1}^{2m+1} V\left(P_i\right)\right|$$

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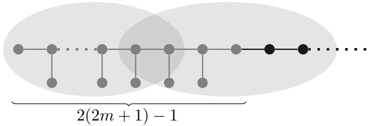
- if X can be partitioned into n triples (whose elements add up to S):

  $\rightarrow$ light central spine vertex of $G_{(2m+1)-i+1}$ in step i (for $1 \leq i \leq m+1$)

  $\rightarrow$ light central vertex of $P_{2(2m+1-i)+1}$ in step i (for $m+2 \leq i \leq 2m+1$)

  $\Rightarrow b(G) \leq 2m+1$

- assume $b(G) = 2m + 1$ and let $(x_1, \ldots, x_{2m+1})$ be an optimal burning sequence:
  - $\rightarrow$ x_i is a spine vertex for all $1 \leq i \leq 2m + 1$
  - $\rightarrow$ burning circles are pairwise disjoint
  - $\rightarrow$ largest burning circle covers G_{2m+1} with spine $P_{2(2m+1)-1}$



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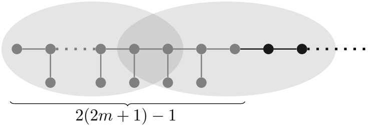

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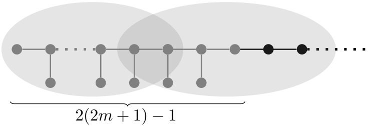

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- universe of DISTINCT 3-PARTITION:
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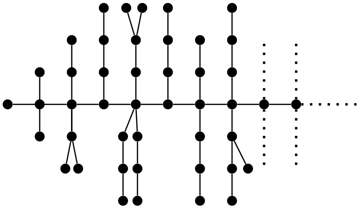
Definition (*p*-Caterpillar)

A *p*-caterpillar G is a tree in which all vertices are within the distance p of a central spine $P_\ell = \{v_1, \ldots, v_\ell\}$, which is the longest path in G.



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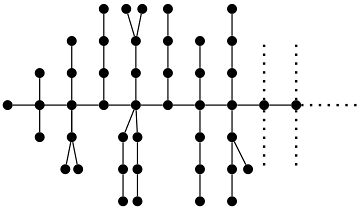

Observation

For a p-caterpillar G we have $b(G) \leq \lceil \sqrt{\ell} \rceil + p$.
Thus, the conjecture is proven to be true for $\lceil \sqrt{n} \rceil \geq \lceil \sqrt{\ell} \rceil + p$.

Theorem (Burning Number Conjecture for $p = 2$)

The burning number of a 2-caterpillar G satisfies $b(G) \leq \lceil \sqrt{n} \rceil$.

Theorem

Let G be a p-caterpillar with at least $2\left\lceil\sqrt{n}\right\rceil - 1$ disjoint legs of length p, then

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- $b(G) > \lceil\sqrt{n}\rceil =: k$ and $|L_p| \geq 2k - 1$ $\quad$ $(L_p$: leaves with distance p to the central spine$)$
- $G - L_p$ is a $(p-1)$-caterpillar:

$$b(G - L_p) \leq \left\lceil\sqrt{n - 2k + 1}\right\rceil \leq \left\lceil\sqrt{n - 2\sqrt{n} + 1}\right\rceil = \lceil\sqrt{n}\rceil - 1$$

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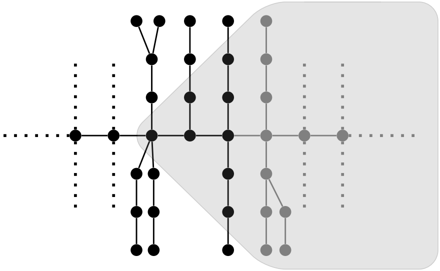
- $G - L_p$ burns after $k - 1$ steps $\Rightarrow$ G burns after k steps $\quad$ ↯

Outlook - Conjecture for p-Caterpillars

- It remains to prove the Burning Number Conjecture for p-caterpillars with $p \geq 3$, less than $2\left\lceil\sqrt{n}\right\rceil - 1$ disjoint legs of length p, and $\left\lceil\sqrt{n}\right\rceil < \left\lceil\sqrt{\ell}\right\rceil + p$.



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* No animals were harmed while researching this topic.