# On List $k$-Coloring Convex Bipartite Graphs

J. Diaz[1], Ö. Y. Diner[1,2], M. Serna[1], O. Serra[1]

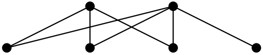
[1]Universidad Politecnica de Catalunya, Barcelona
[2]Kadir Has University, İstanbul

15 September 2020

## Outline

# COL: Coloring

- Given a graph $G = (V, E)$ and a non-negative integer $k$.



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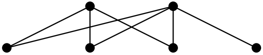

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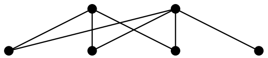

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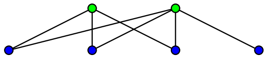
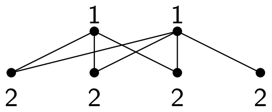
- A *k-coloring* is a map $c : V \to [k]$.





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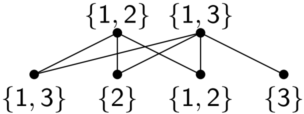
- A *list assignment* $L : V \to 2^{\mathbb{N}}$.



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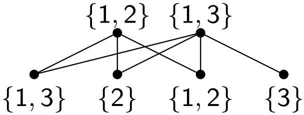
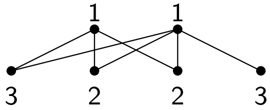



- List Coloring (LiCol) (Erdős 1979, Vizing 1976) asks for the existence of a proper coloring $c$ that obeys $L$, i.e., $\forall v \in V, c(v) \in L(v)$. If it exists, $G$ is said to be *$L$-colorable*.



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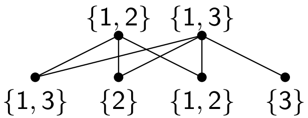


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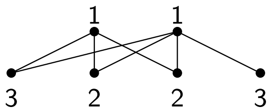


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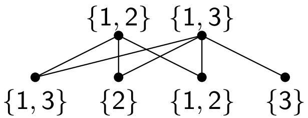


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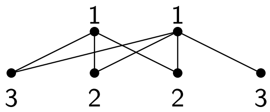


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- $k$-List Coloring ($k$-LiCol) each list $L$ has size at most $k$.

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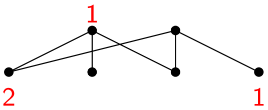

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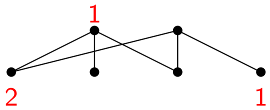


- In $k$-Precoloring Extension ($k$-PrExt) total number of colors is bounded by $k$.

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- LI $H$-COL is a generalization of LI $k$-COL.

## Relationships Between Coloring and its Variants

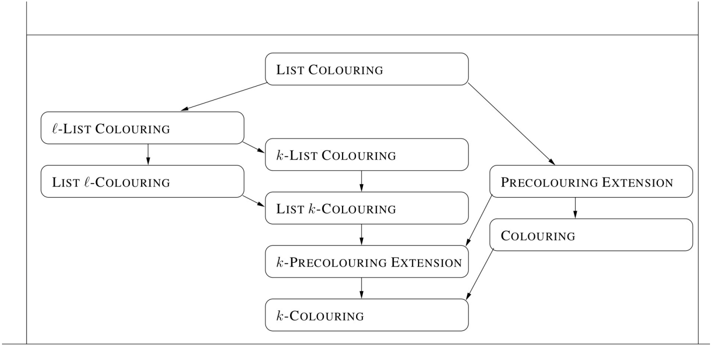
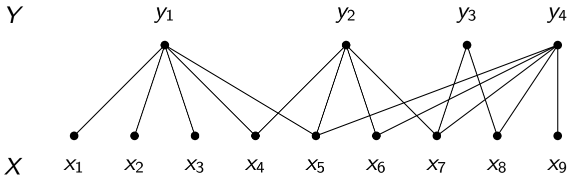



Figure from Golovach, Johnson, Paulusma and Song 2014.

## Bipartite Graphs: Adjacency and Enclosure Properties

- Given a bipartite graph $G = (X \cup Y, E)$, $X$ and $Y$ form a bipartition of the vertex set into stable sets.



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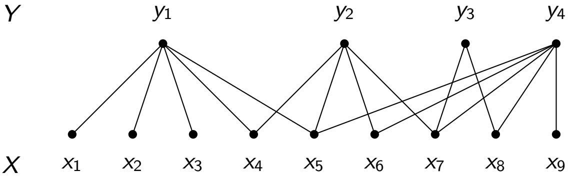


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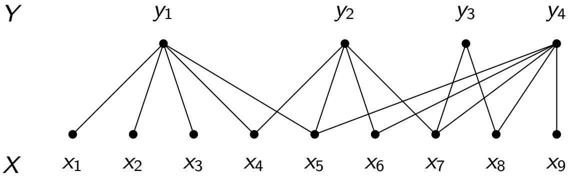


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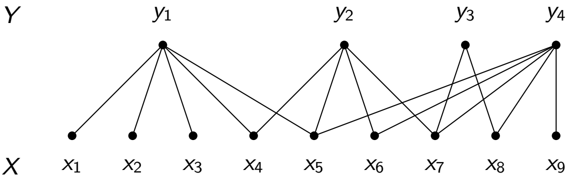
- An ordering for $X$ has the *enclosure property* if for every pair of vertices $u, v \in Y$ s. t. $N(u) \subseteq N(v)$, the vertices in $N(v) \backslash N(u)$ occur consecutively in the ordering of $X$.

- *Chordal bipartite graphs* are bipartite graphs s. t. every cycle $C_n$ of length at least 6 has a chord.



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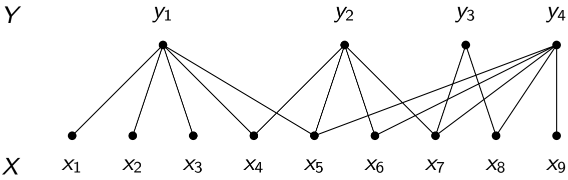

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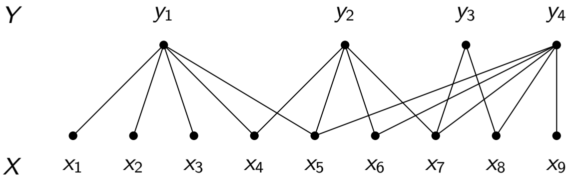

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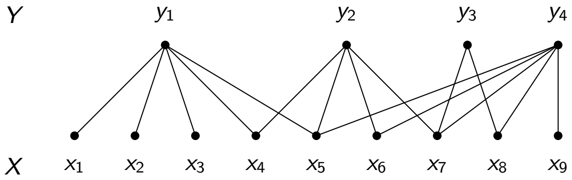
- *Bipartite permutation graphs* are biconvex bipartite graphs in which one of the partite sets obeys both the adjacency and the enclosure properties.



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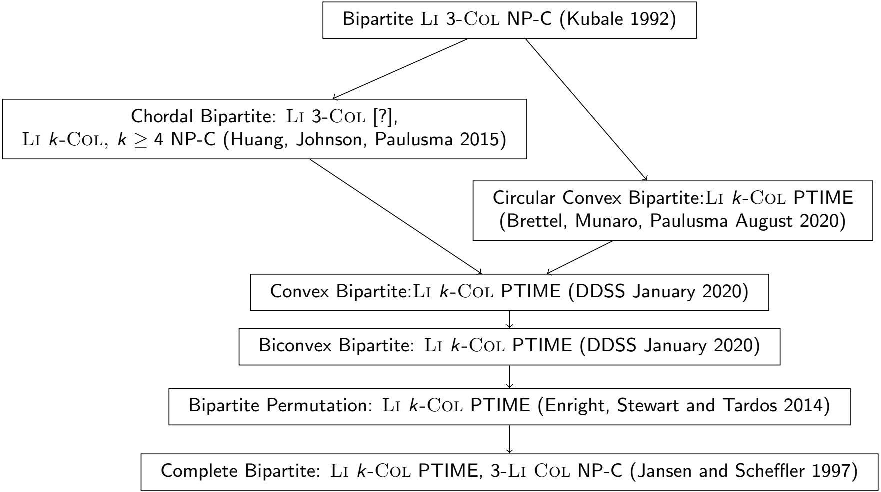
- Surveys: Tuza 1997, Paulusma 2016.

# Chart


Bipartite Li 3-Col NP-C (Kubale 1992)
Chordal Bipartite: Li 3-Col [?],
Li $k$-Col, $k \geq 4$ NP-C (Huang, Johnson, Paulusma 2015)
Circular Convex Bipartite:Li $k$-Col PTIME
(Brettel, Munaro, Paulusma August 2020)
Convex Bipartite:Li $k$-Col PTIME (DDSS January 2020)
Biconvex Bipartite: Li $k$-Col PTIME (DDSS January 2020)
Bipartite Permutation: Li $k$-Col PTIME (Enright, Stewart and Tardos 2014)
Complete Bipartite: Li $k$-Col PTIME, 3-Li Col NP-C (Jansen and Scheffler 1997)

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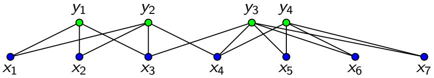

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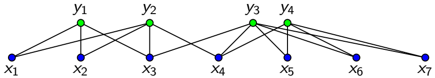
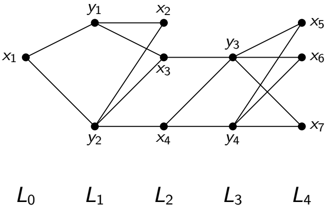



- The *distance layers* of a connected graph $G = (V, E)$ from a vertex $v_0$ are $L_0, L_1, ..., L_z$, where $L_0 = \{v_0\}$ and, for $i > 0$, $L_i$ consists of the vertices at distance $i$ from $v_0$ and $z$ is the largest integer for which this set is non-empty.



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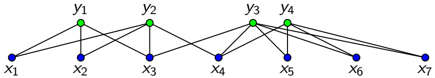


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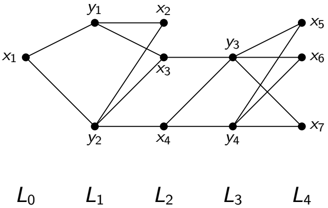


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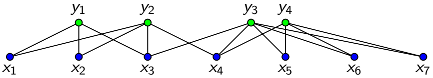


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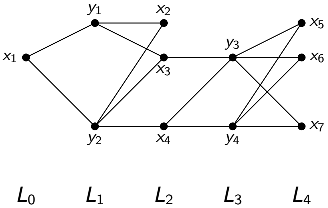


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- Chain graphs are a proper subclass of convex bipartite graphs.

**Theorem (Enright, Stewart and Tardos 2014)**

- *For a fixed graph $H$, Li $H$-Col is solvable in polynomial time for $G$ s. t. every connected induced subgraph of $G$ admits a multichain ordering.*
- *All connected bipartite permutation graphs and interval graphs admit multichain orderings.*

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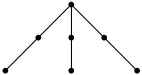
*If $G$ is a biconvex graph, then $G$ does not contain subd$(K_{1,3})$ as an induced subgraph.*



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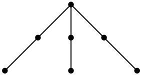
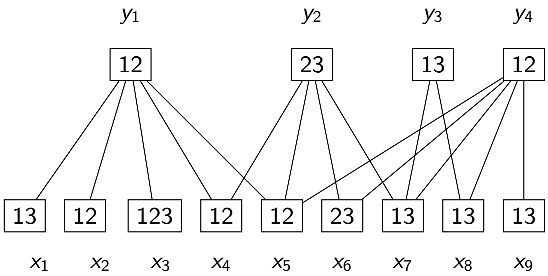


### Theorem (DDSS, 2020)

- *Every connected biconvex graph admits a multichain ordering.*
- *For any $H$,* LI $H$-COL *is solvable in polynomial time when restricted to biconvex graphs.*
- LI $k$-COL *and* $k$-PREXT *are solvable in polynomial time when restricted to biconvex graphs.*

## Color Algorithm




• Let $G = (X \cup Y, E)$ be a connected bipartite graph that is convex w.r.t. $X = \{x_1, \ldots, x_n\}$, thus for each $y \in Y$ there are two positive integers $a_y \leq b_y$ s. t. $N(y) = \{x_i \mid a_y \leq i \leq b_y\}$.

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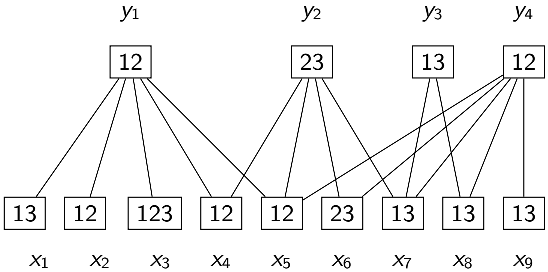


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- Define $A := \{a_y \mid y \in Y\}$ and $B := \{b_y \mid y \in Y\}$.

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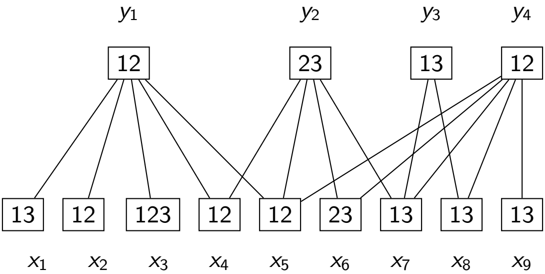


- Define $A := \{a_y \mid y \in Y\}$ and $B := \{b_y \mid y \in Y\}$.

$A = \{1, 4, 5, 7\}$ and $B = \{5, 7, 8, 9\}$.

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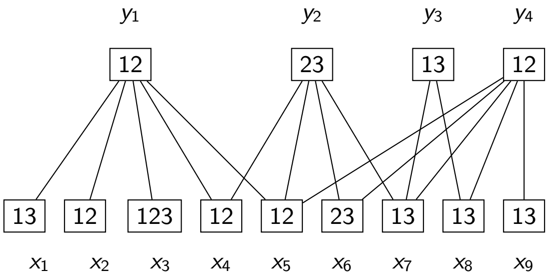


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• Assume that $B = \{b_1, \ldots, b_\beta\}$ are sorted so that $b_1 < b_2 < \cdots < b_\beta$. By the connectivity of $G$, $b_\beta = n$.

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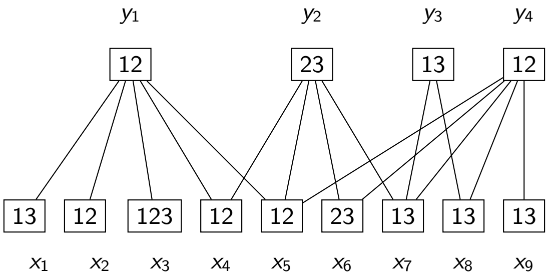


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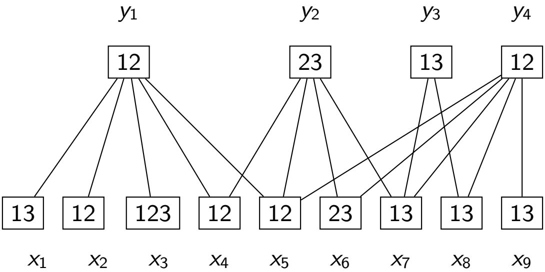


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$b_2 = 7, X_2 = \{x_1, x_2, ..., x_7\}$, $Y_2 = \{y_1, y_2\}$ and $Z_2 = \{y_3, y_4\}$.

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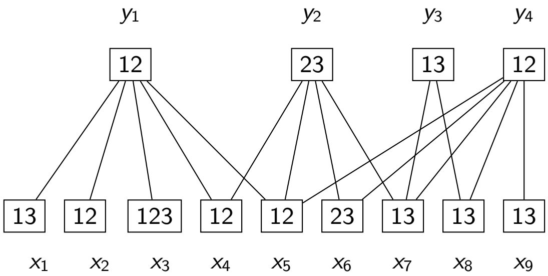


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- $Z_j$ contains vertices in $Y$ whose neighborhood starts before or at $b_j$ and ends after $b_j$.

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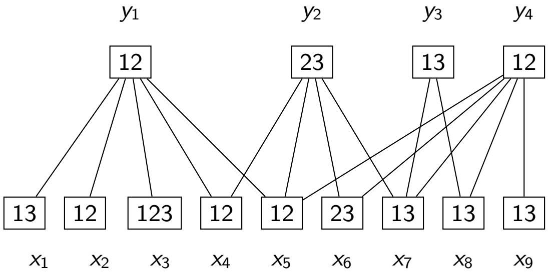


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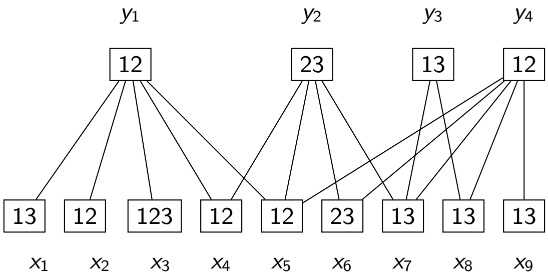


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- Fix $j$, $1 \leq j \leq \beta$, define $A(j) = \{a_y \mid y \in Z_j\} \cup \{b_j\}$ and order it increasingly $\{a_{1,j}, \ldots, a_{\alpha_j,j} = b_j\}$.

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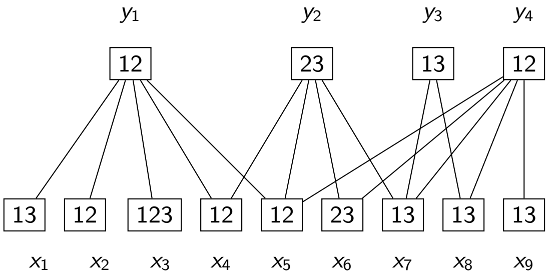


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- For each $1 \leq i \leq \alpha_j$ and $S \subsetneq [k]$, $T_j(i, S)$ will hold value $\mathrm{TRUE}$ whenever there is a valid list coloring of $G_j$ s. t. it uses no color in $S$ for the set $X_i^j = \{x_\ell \mid a_{i,j} \leq \ell < a_{i+1,j}\}$.

## What does the color algorithm do?

### The Color Algorithm

- In going from $j-1$ to $j$, compute the values for the $x \in X_j$ that were not in $X_{j-1}$ combining this information with the relevant information computed in the previous step.
- Next, incorporate the restriction from $y \in Y_j$ that were not in $Y_{j-1}$.
- Finally, rearrange the information to keep only the values for the index in $A(j)$.

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**Lemma (DDSS, 2020)**

*Let $G = (X \cup Y, E)$ be a connected convex bipartite graph, $L$ be a color assignment for $G$. There is an $L$–coloring of $G$ if and only if there is $S \subseteq K$ such that at the end of the execution of the Color Algorithm $T_\beta(\alpha_\beta, S) = \mathrm{TRUE}$.*

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**Theorem (DDSS, 2020)**

- *For $k \geq 3$, LI $k$-COL and $k$-PREXT on convex bipartite graphs can be solved in polynomial time.*
- *For any $H$, LI $H$-COL on convex bipartite graphs can be solved in polynomial time.*

## Open Problems

- What is the compuational complexity of Li 3-Col when restricted to chordal bipartite graphs?

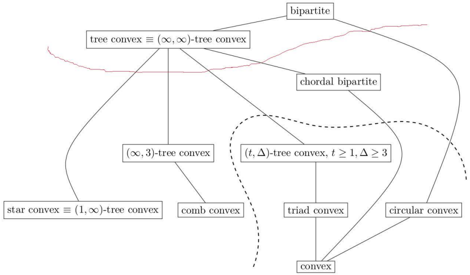
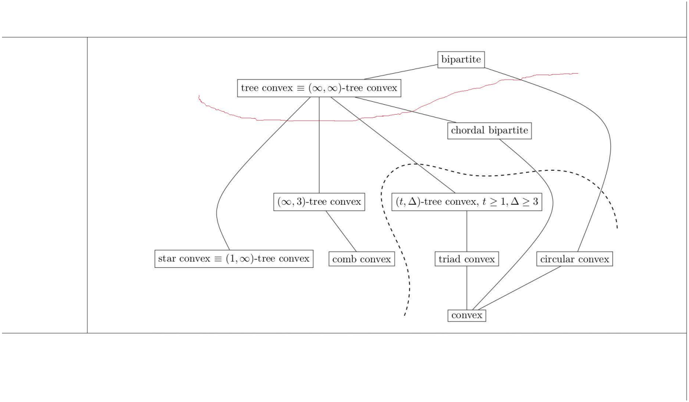



Figure -modified - from Brettel, Munaro, Paulusma 2020.

## Open Problems

- What is the compuational complexity of Li 3-Col when restricted to chordal bipartite graphs?
- .. Li $k$-Col for $k \geq 3$ when restricted to star convex graphs or to comb convex graphs?

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Thank you for your attention.