# Improved Bounds on the Span of $L(1, 2)$-edge Labeling of Some Infinite Regular Grids

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# Outline

## $L(h,k)$-vertex labeling:

- It is a labeling $f : V \rightarrow \{0, 1, \cdots, n\}$ of a graph $G = (V, E)$.
- $|f(u) - f(v)| \geq h$, if $d(u, v) = 1$,
- $|f(u) - f(v)| \geq k$, if $d(u, v) = 2$,
- Span $\lambda_{h,k}(G)$ is minimum $n$ for $L(h,k)$-vertex labeling in $G$.

## $L(h,k)$-edge labeling:

- It is a labeling $f : E \rightarrow \{0, 1, \cdots, n\}$ of a graph $G(V, E)$.
- $|f'(e_1) - f'(e_2)| \geq h$, if $d(e_1, e_2) = 1$,
- $|f'(e_1) - f'(e_2)| \geq k$, if $d(e_1, e_2) = 2$,
- Span $\lambda'_{h,k}(G)$ is minimum $n$ for $L(h,k)$-edge labeling in $G$.

## Frequency Channel Assignment Problem(CAP):

- CAP can be formulated as a $L(h,k)$-vertex(edge) labeling problem In Infinite regular grids.
- $\lambda_{h,k}(G)$ or $\lambda'_{h,k}(G)$ has practical relevance.

### Main results

- We improve $\lambda'_{1,2}(G)$ for infinite hexagonal grid($T_3$), square grid($T_4$) and triangular grid($T_6$).
- $\lambda'_{1,2}(T_3) = 7$(Previously $7 \leq \lambda'_{1,2}(T_3) \leq 8$, Lin. W. and Wu. J).
- $\lambda'_{1,2}(T_4) = 11$(Previously $10 \leq \lambda'_{1,2}(T_4) \leq 11$, Lin. W. and Wu. J).
- $\lambda'_{1,2}(T_6) \geq 19$(Previously $16 \leq \lambda'_{1,2}(T_6) \leq 20$, Calamoneri. T).

**Line graph($L(G)(V', E')$) of a graph $G(V, E)$ :**

- Edges of $G$ represent vertices in $L(G)$.
- Edge exists in $L(G)$ if corresponding edges in $G$ have common vertex.
- $\lambda'_{h,k}(G)$=$\lambda_{h,k}L(G)$.
- We derive the bounds for $T_3$ and $T_4$ using $L(T_3)$ and $L(T_4)$.
- For $T_6$ we derive the bound in $T_6$ directly.

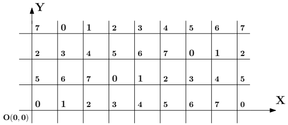



**Figure 1 :** A $L(1,2)$-vertex labeling of $L(T_3)$.

## Theorem 1

$\lambda'_{1,2}(T_3) = 7.$

## Proof.

- $g(v)_{(x,y)} = (x + 5y)$ *mod* 8 is a coloring function for vertices $v_{(x,y)}$ at $L(T_3)$( Figure 1).
- $\lambda'_{1,2}(T_3) = \lambda_{1,2}(L(T_3)) \leq 7$.
- $\lambda'_{1,2}(T_3) \geq 7$(Lin. W and Wu. J).
- Hence, $\lambda'_{1,2}(T_3) = \lambda_{1,2}(L(T_3)) = 7$.

□

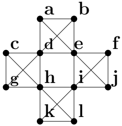



**Figure 2 :** Sub graph $G$ of $L(T_4)$.

- $d$, $e$, $h$, $i$ are **central vertices** in $G$(Figure 2).
- $a$, $b$, $c$, $f$, $g$, $j$, $k$, $l$ are **peripheral vertices.**

### Observation 1

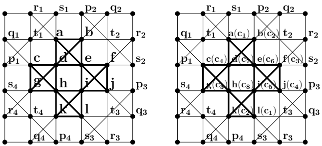
*If $f(x) = f(y) = c$ (Either $x \in \{a, b\}$, $y \in \{k, l\}$ or $x \in \{c, g\}$, $y \in \{f, j\}$ and $c$ be a non-extreme color) then $c \pm 1$ can only be used in Either $\{a, b, k, l\} \setminus \{x, y\}$ or in $\{c, g, f, j\} \setminus \{x, y\}$.*

## Observation 2

*If $f(x) = f(y) = c$ and $|f(x) - f(u)| \geq 2$(Either $\{x, u\} = \{a, b\}$, $y \in \{k, l\}$ or $\{x, u\} = \{c, g\}$, $y \in \{f, j\}$ and $c$ be a non-extreme color) then $c + 1$ or $c - 1$ remain unused in $G$.*

## Theorem 2

$\lambda_{1,2}(L(T_4)) \geq 11$.




a. b.

**Figure 3 :** Sub graph $G_1$ of $L(T_4)$

## Proof

- If all vertices of $G$ have distinct colors:
  $G$ have 12 vertices,$\lambda'_{1,2}(T_4) = \lambda_{1,2}(L(T_4)) \geq \lambda_{1,2}(G) \geq 11$.
- At most a pair of peripheral vertices have same color in all sub graphs isomorphic to $G$:
  - $f(a) = f(l)$, $f(d) = f(a) + n, n \geq 2$(Figure 3.a).
  - Reusing $x \in \{f(a), f(a) + n\}$ in $G'_2$(Central vertices $\{b, e, f, t_2\}$) results $\lambda_{1,2}(G'_2) \geq 11$(Observation 2).
  - Otherwise $\lambda_{1,2}(G') \geq 11$.
- At least one sub graph of $L(T_4)$ isomorphic to $G$ where two pair of peripheral vertices have same color:
  - $f(a) = f(l)$, $f(c) = f(j)$ or $f(a) = f(l)$, $f(b) = f(k)$.(Figure 3.a).
  - Using Observation 1 and Observation 2, $\lambda_{1,2}(G_1) \geq 11$.

- At least one sub graph of $L(T_4)$ isomorphic to $G$ where three pair of peripheral vertices have same color:
  - $f(a) = f(l)$, $f(b) = f(k)$, $f(c) = f(j)$(Figure 3.a).
  - Using Observation 1 and Observation 2, $\lambda_{1,2}(G_1) \geq 11$.
- At least one sub graph of $L(T_4)$ isomorphic to $G$ where four pair of peripheral vertices have same color:
  - $f(a) = f(l) = c_1$, $f(b) = f(k) = c_2$, $f(g) = f(f) = c_3$, $f(c) = f(j) = c_4$(Figure 3.b).
  - From Observation 1, Observation 2 and re usability of $c_1$, $c_2$, $c_3$ and $c_4$, $\lambda_{1,2}(G_1) \geq 11$.

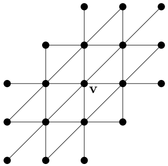



**Figure 4 :** A subgraph $G_v(V, E)$ of $T_6$.

**Set of edges $S_1$, $S_2$ and $S_3$ are defined below.**

- $S_1$: All edges $e \in E$(Figure 4) where $e$ incident to $v$.
- $S_2$: All edges $e \in E$ where end points of $e$ are incident to $e_1 \in S_1$ and $e_2 \in S_1$.
- $S_3$: $E \setminus (S_1 \cup S_2)$.

**Preliminary results for $L(1,2)$-edge labeling at $G_v$:**
Let $f'(e) = c$ where $e \in E$.

- $\forall e' \in E \setminus e$, $f'(e') \neq c$ if $e \in S_1$;
- If $c$ is used in $S_2(S_3)$ then $c$ can be used atmost twice(thrice) in $G$.
- If $c$ is used in $S_1/S_2/S_3$ then both $c+1$ and $c-1$ can be used atmost once/twice/thrice in $G$.
- At least 6, 3 and 6 colors are required for $S_1$, $S_2$ and $S_3$ respectively.

**Lemma 3**

*For optimal labeling of $G_v$, $S_1$ gets 6 consecutive colors including minimum or maximim color.*

**Proof.**

- Optimal labeling must use 6 consecutive colors in $S_1$.
- Optimal labeling use Minimum(*min*) or maximum(*max*) color in $S_1$ as $min - 1$ or $max + 1$ does not exists.

□

**Theorem 4**

*For Optimal labeling of $G_v$, $\{c, c+2, c+4\}$ must be used in $S_2$.*

**Proof.**

- $C_{S_2}$ set of all colors $c-1$, $c$ and $c+1$( $f'(e)=c$, $e \in S_2$).
- Cardinality of $C_{S_2} \geq 6$ when $\{c, c+2, c+4\}$ used in $S_2$.
- Cardinality of $C_{S_2} \geq 7$ otherwise.

□

## Lemma 5

*If $c$, $c+1$, $c+2$ used thrice in $S_3$, then $c-1$ and $c+3$ can not be used in $S_3$.*

## Proof.

- $H$ is the set of vertices incident to edges of $S_2$.
- Color $c$ can be used thrice in any one of two sets of three alternating vertices in $H$.
- If $e_1, e_2 \in S_3$, $f(e_1) = c$, $f(e_2) = c-1$ then $\exists e_3 \in S_2$ such that $e_1, e_2, e_3$ form a triangle.
- $\exists e_1, e_2 \in S_3$ where $f(e_1) = c$, $f(e_2) = c-1$, $d(e_1, e_2) = 2$ leads a contradiction for $c-1$ used thrice.
- Same holds for $c+2$ and $c+3$.

□

## Theorem 6

$\lambda'_{1,2}(G_v) \geq 17$.

## Proof.

- Let $S_1$ uses minimum color and needs 6 colors(lemma 3).
- Let $S_2$ uses $c$, $c+2$ and $c+4$ each twice(Theorem 4).
- $S_1$ uses colors $\{c-2, \cdots, c-7\}$.
- $c+1$, $c+3$, $c+5$ are used twice each in $S_3$.
- Remaining 12 edges are colored by at least 5 colors, as no 4 consecutive colors can be used thrice in $S_3$(lemma 5).
- $\lambda'_{1,2}(G_v) \geq (c+10)-(c-7) = 17$.

□

- $\exists v' \in V$ for which minimum and maximum colors are not used in edges incident to $v'$. $G_{v'}$ is isomorphic to $G_v$ centering $v'$.
- $min'$ and $max'$ be minimum and maximum color used in $S_1'$.

**Lemma 7**

$|max' - min'| \geq 7$ *results* $\lambda'_{1,2}(G_{v'}) \geq 19$.

**Proof.**

- two colors $c_1, c_2$ are unused in $S_1'$.
- $c_1, c_2, min' - 1, max' + 1$ be used at most once in $G_{v'} \setminus S_1'$.
- They must be used at least twice in $G_{v'} \setminus S_1'$.
- To color the 4 edges, 2 additional colors are required resulting $\lambda'_{1,2}(G_{v'}) \geq 19$.

□

## Theorem 8

$\lambda'_{1,2}(T_3) \geq 19.$

## Proof.

*u* and *w* are in *H*.

1. *u*, *w* are connected by an edge.
   - Consecutive colors can be given to edges incident to both of *u* and *w*.
   - $|max - min| \geq 7$ (*min*, *max* are minimum and maximum colors used in $S_1$) results $\lambda'_{1,2}(G_v) \geq 19$(lemma 7).
2. *u*, *w* are not connected by an edge but have a common neighbour.
   - Similar argument holds as previous one.

□

## Future works

- To Determine $\lambda'_{h,k}(G)$ for $T_3$, $T_4$ and $T_6$ for other values of $h$ and $k$.
- To Determine $\lambda'_{h,k}(G)$ for other graph classes.