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Recognizing Cartesian products of matrices and polytopes

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Decomposing an object as a product of simpler objects:
a ubiquitous theme.

- Factorization of a number into prime factors
- Graph decomposition (perfect graphs, ...)
- Matroids (Seymour's decomposition, ...)
- Matrix factorization (LU, LDU, rank, ...)
- Polytopes (Cartesian product)

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- Polytopes (Cartesian product)

Goal: efficiently decompose into irreducible factors.

Outline

- The operation of 1-product between matrices closely follows the Cartesian product of polyhedra.
- We give a polynomial time algorithm to recognize and decompose 1-products.
- We apply our result to the slack matrix recognition problem.

The **1-product** of matrices $A \in \mathbb{R}^{m_1 \times n_1}$ and $B \in \mathbb{R}^{m_2 \times n_2}$ is the matrix $A \otimes B$ in $\mathbb{R}^{(m_1+m_2)\times(n_1 n_2)}$ whose columns are the concatenation of each column of A with each column of B.

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 2 & 3 & 3 & 3 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix}$$

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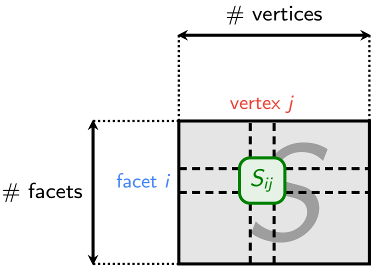
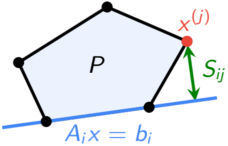
We will consider the following recognition problem:
given a matrix S, is S equal to $A \otimes B$ for some A, B, up to permutation of rows and columns?

Slack matrix

Polytope $P = \operatorname{conv}(\{x^{(1)}, \dots, x^{(m)}\}) = \{x \in \mathbb{R}^d \mid Ax \leqslant b\}$.





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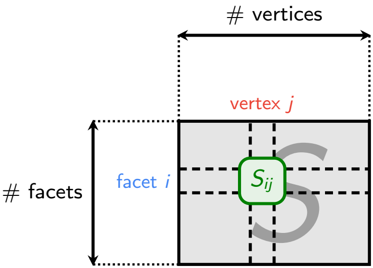

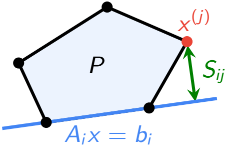

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Slack matrix recognition: given a matrix S, determine whether S is the slack matrix of some polytope.

Equivalent to polyhedral verification, whose complexity is unknown. (Gouveia, Grappe, Kaibel, Pashkovich, Robinson, Thomas 2013)

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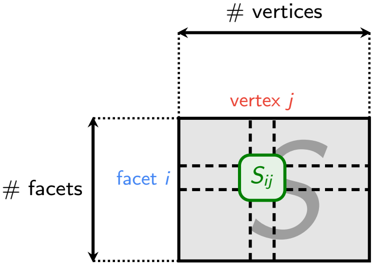

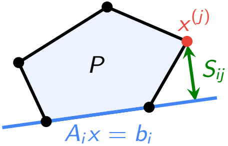

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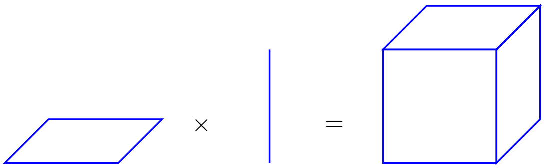
Slack matrix recognition: given a matrix S, determine whether S is the slack matrix of some polytope.

Conjecture

Slack matrix recognition is polynomial for 0/1 matrices (**2-level polytopes**).

Cartesian product and 1-product

Given polytopes $P_1 \subseteq \mathbb{R}^{d_1}$, $P_2 \subseteq \mathbb{R}^{d_2}$, we have
$P_1 \times P_2 := \{(x_1, x_2) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \mid x_1 \in P_1,\ x_2 \in P_2\}$.



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Observation: Let $S = S_1 \otimes S_2$, then:
S is the slack matrix of a polytope $P \iff$ there exist polytopes P_i, $i \in \{1, 2\}$ such that S_i is the slack matrix of P_i and $P \approx P_1 \times P_2$.

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It is also equivalent to recognizing the list of vertices of a Cartesian product (put in matrix form).

Is this matrix a 1-product?

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 2 & 1 & 2 & 1 \end{pmatrix}$$

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$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 2 & 1 & 2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \end{pmatrix}$$

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We say that the matrix is a 1-product with respect to the above row partition. Once we guess this partition, it is easy to identify $A \otimes B$.

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$$\begin{pmatrix} 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 2 \end{pmatrix}$$

No! The first and the second row can't belong to different parts of the row partition, and similarly for the second and third row.

Main result

Theorem

Given a matrix S, there is a polynomial time algorithm that determines whether S is a 1-product and, in case it is, outputs the factors A, B such that $S = A \otimes B$.

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Idea: reduce the problem to (symmetric) submodular minimization, and solve via Queyranne's algorithm.

Intuition

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 2 & 3 & 3 & 3 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Imagine to pick a column uniformly at random.

$$\Pr \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} = \Pr \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \Pr \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The color of the first half is **independent** from the color of the second half.

Information theory

Let $A \in \mathcal{A}$ and $B \in \mathcal{B}$ be discrete random variables.

The **mutual information** of A and B is:

$$I(A; B) = \sum_{a \in \mathcal{A}, b \in \mathcal{B}} \Pr(A = a, B = b) \cdot \log_2 \frac{\Pr(A = a, B = b)}{\Pr(A = a) \cdot \Pr(B = b)}$$

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Property: $I(A; B) \geq 0$, and $I(A; B) = 0 \iff A$ and B are independent.

Proof of the main result

Let $(X, \overline{X})$ any partition of the row set of our matrix S. Let C be a uniformly chosen column of S, and C_X its restriction to X. Define:

$$f(X) := I(C_X; C_{\overline{X}})$$

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- If $f(X) = 0$, S is a 1-product (and we can get factors A, B)
- If $f(X) > 0$, S is not a 1-product!

Theorem

Given a matrix S, there is a polynomial time algorithm that determines whether S is a 1-product and, in case it is, outputs the factors A, B such that $S = A \otimes B$.

Theorem

Let S be a matrix whose columns are all distinct. Then S has a "unique" decomposition $S = S_1 \otimes \cdots \otimes S_t$, where each S_i is irreducible.

Proof: the family of minimizers of a submodular function is closed under intersection...

Theorem

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Theorem

Given S slack matrix of P, there is a polynomial time algorithm that determines if P is **affinely equivalent** to a Cartesian product $P_1 \times P_2$ and, in case it is, outputs two matrices S_1, S_2 such that S_i is the slack matrix of P_i.

Note: determining if P is **equal** to a Cartesian product is a much easier problem!

Second result

Conjecture

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Given a 0/1 matrix S, there is a polynomial algorithm that correctly determines if S is the slack matrix of a 2-level **matroid base** polytope.

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Given a 0/1 matrix S, there is a polynomial algorithm that correctly determines if S is the slack matrix of a 2-level **matroid base** polytope.

- 2-level matroids can be decomposed into uniform matroids through 1-sums (related to 1-products) and 2-sums (Grande and Sanyal 2016).

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- We define a notion of 2-product related to 2-sums, and extend our results to 2-products.
- We decompose the slack matrix into "simple" pieces using our algorithm, and reconstruct the original matroid.

Conclusion

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Thank you for your attention!