Range-Relaxed Graceful Game

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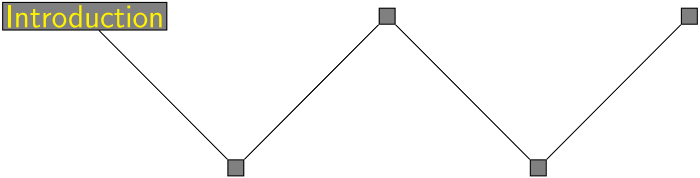
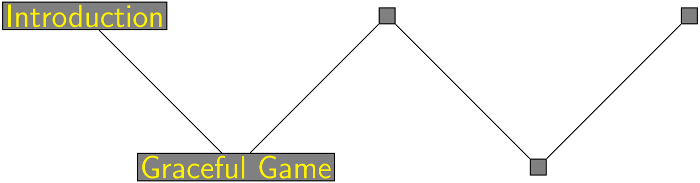
Summary


Introduction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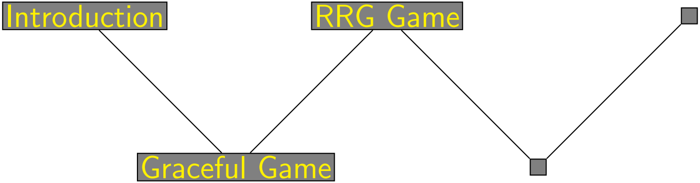
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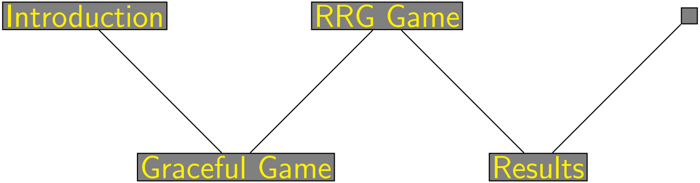
RRG Game



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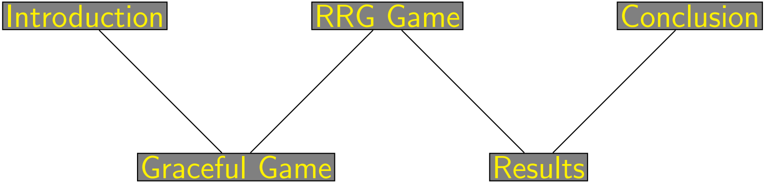
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RRG Game
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Graceful Game
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Introduction

Graph Labeling

Area of graph theory whose main concern consists in determining the feasibility of assigning labels to the elements of a graph satisfying certain conditions.

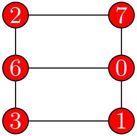

Figure: $P_2 \square P_3$.

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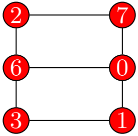

Figure: $P_2 \square P_3$.

- We are interested in a vertex labeling called graceful labeling and its relaxed version.

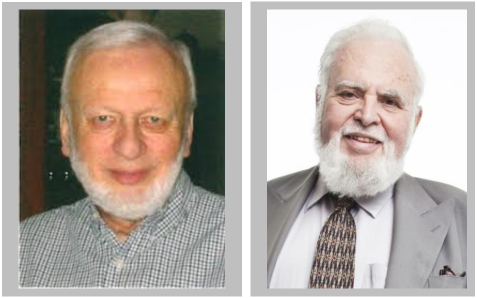

Figure: **A. Rosa and S. Golomb**

Graceful Tree Conjecture (Rosa, 1966)

All trees are graceful.

Range Relaxed Graceful Labeling

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- A vertex labeling $f : V(G) \longrightarrow \mathscr{L}$ that induces an edge labeling $g : E(G) \longrightarrow Z$ is called range-relaxed graceful labeling if:
  (i) f is injective;
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  $g(uv) = | f(u) - f(v) |$, then all edge labels are distinct.

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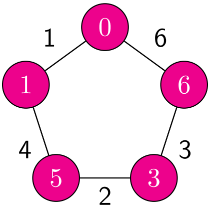

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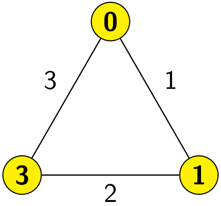

Figure: **Z. Tuza**

Labeling Games

The *Graceful Game* emerged in 2017, first proposed by Tuza.

Combinatorial Games

Definition

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- there is no randomness.

The Graceful Game

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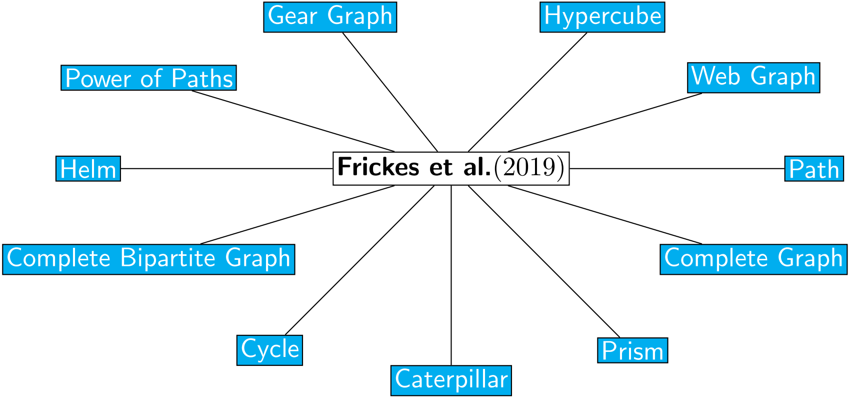
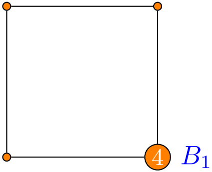
- The game ends if there is no legal move possible or a graceful labeling of G is created.
- Alice wins if G is gracefully labeled and Bob wins if he can prevent it from happening.

Previous Results on Graph Classes



Lemma 1 (Frickes et al., 2019)

Let G be a simple graph. In any step of the graceful game, Alice can only use the label 0 (resp. m) to label a vertex $v \in V(G)$ if v is adjacent to every remaining vertex not yet labeled or v is adjacent to a vertex already labeled by Bob with m (resp. 0).



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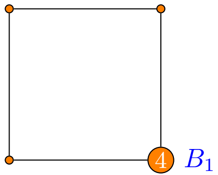
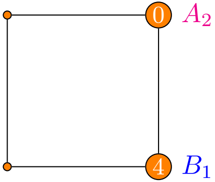
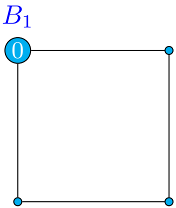




Lemma 2 (Frickes et al., 2019)

Let G be a simple graph. If Bob assigns 0 (resp. m) to a vertex $v \in V(G)$, where v has two non-adjacent vertices or only one adjacent vertex, then Alice is forced to label a vertex adjacent to v with m (resp. 0).



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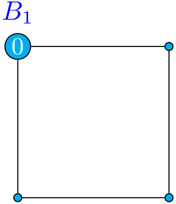

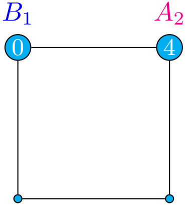

Main Results

Toroidal Grid

The toroidal grid graph $T_{p,q}$, with $p, q \in \mathbb{N}$ and $p, q \geqslant 3$, is defined as the cartesian product $C_p \square C_q$ of two cycles C_p and C_q.

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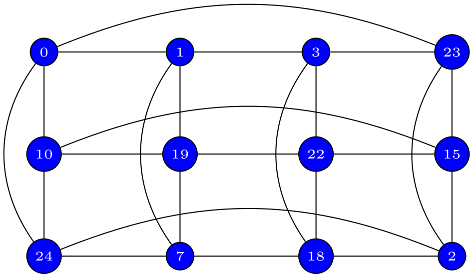

Figure: Graceful labeling of $T_{3,4}$.

Theorem (Jungreis and Reid, 1992)

- $T_{p,q}$ with $p \equiv 0 \pmod 4$ and $q \equiv 0 \pmod 2$ are graceful.
- $T_{p,q}$ with p and q odd are not graceful.

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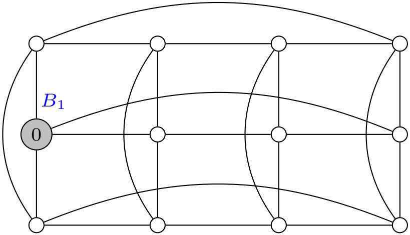
Theorem 6 (Oliveira, Dantas and Luiz, 2020)

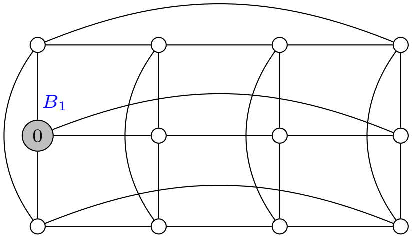
Bob has a winning strategy for all toroidal grids.

Sketch of the Proof of Toroidal Grids

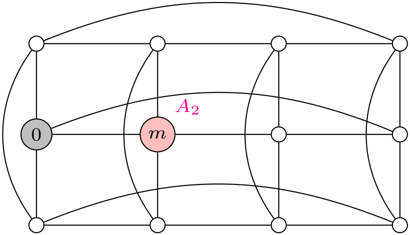
- Let $G \cong C_p \,\square\, C_q$, for $p, q \geqslant 3$.



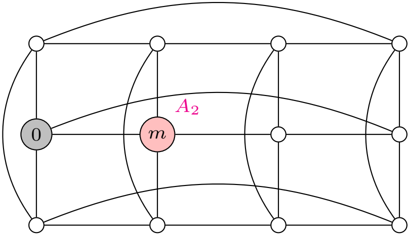
- Let $G \cong C_p \,\square\, C_q$, for $p, q \geqslant 3$.
- Bob is the first player.



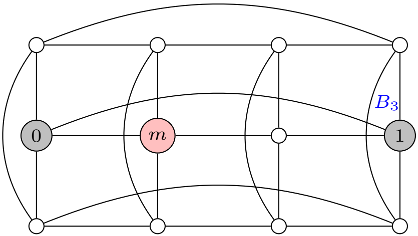
- He assigns label 0 to an arbitrary vertex $v \in V(G)$.



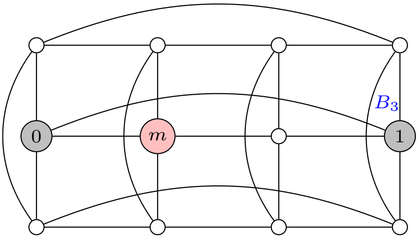
- He assigns label 0 to an arbitrary vertex $v \in V(G)$.
- Alice is forced to assign label m.



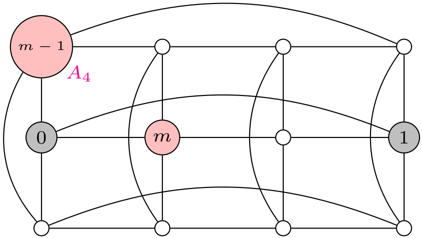
- Bob assigns label 1 to a free neighbour of v.



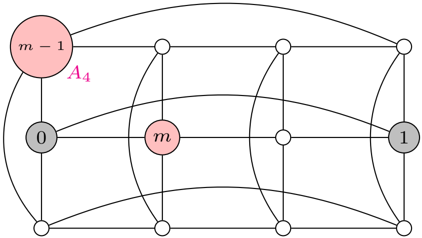
- Bob assigns label 1 to a free neighbour of v.
- Creating the edge label 1.



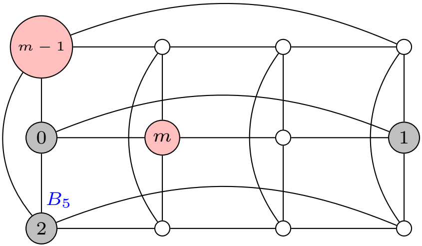
- $\mid m-1 \mid$ or $\mid (m-1)-0 \mid$.



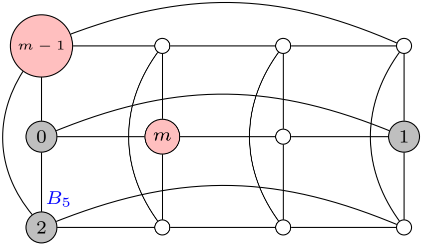
- $\mid m - 1 \mid$ or $\mid (m - 1) - 0 \mid$.
- In the fourth move, Alice assigns label $m - 1$.



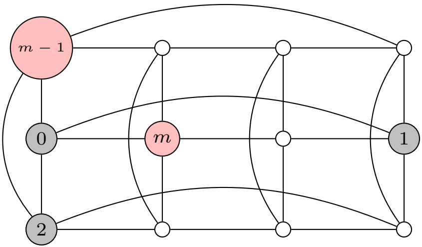
- Bob assigns label 2 to the unique free neighbour of v.



- Bob assigns label 2 to the unique free neighbour of v.
- $\mid m-2 \mid$ or $\mid (m-1)-1 \mid$ or $\mid (m-2)-0 \mid$.

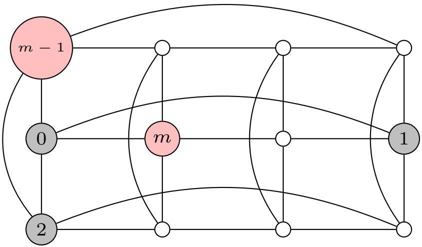


- Bob exhausts Alice's possibilities of creating the edge label $m-2$.

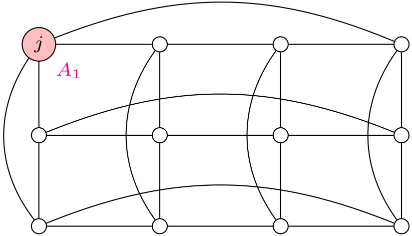


Sketch of the Proof

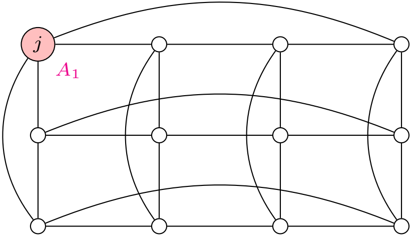
- Bob exhausts Alice's possibilities of creating the edge label $m-2$.
- Bob is the winner.



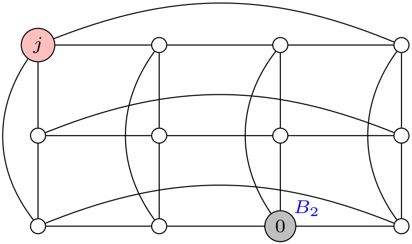
- Alice is the first player.



- Alice is the first player.
- She assigns a label $j \in \{1, \ldots, m-1\}$ to $v \in V(G)$. (Lemma 1)



- We first consider the case where $j \in \{2, \ldots, m-1\}$.



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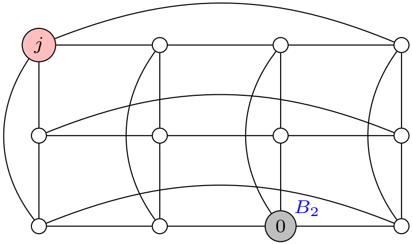
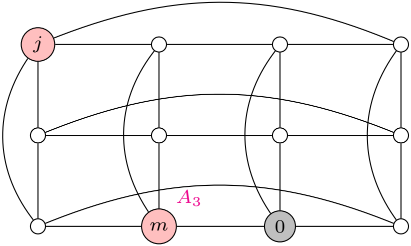
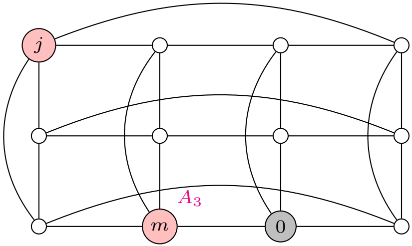
- Bob assigns label 0.



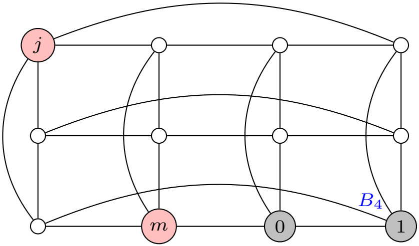
- In the third move, Alice is forced to assign label m.



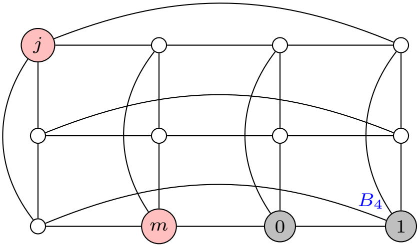
- In the third move, Alice is forced to assign label m.
- Creating the edge label m.



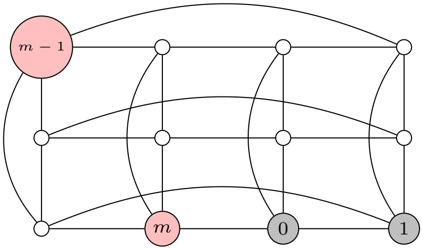
- In the fourth move, Bob assigns label 1 to a free neighbour of u.



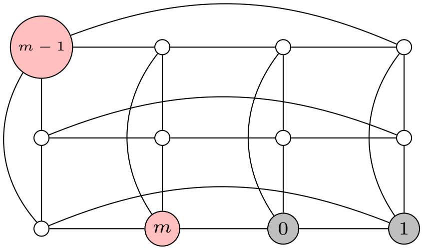
- In the fourth move, Bob assigns label 1 to a free neighbour of u.
- $\mid m - 1 \mid$ or $\mid (m-1) - 0 \mid$.



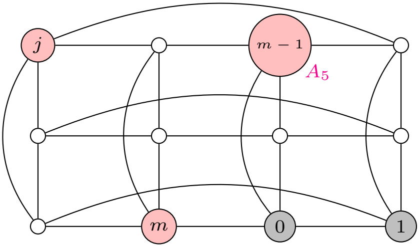
- If $j = m - 1$, then the game is over at this point.



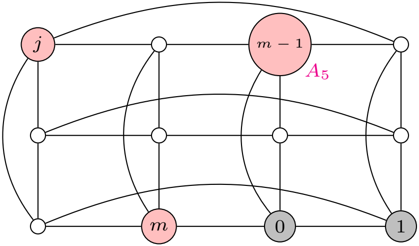
- If $j = m - 1$, then the game is over at this point.
- Bob is the winner.



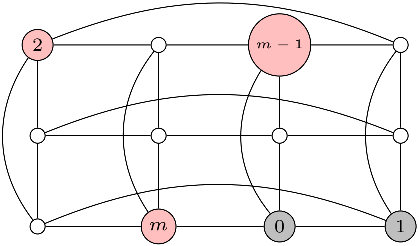
- If $j \neq m-1$, Alice is forced to assign label $m-1$.



- If $j \neq m-1$, Alice is forced to assign label $m-1$.
- Creating the edge label $m-1$.



- If $j = 2$, Bob wins the game.



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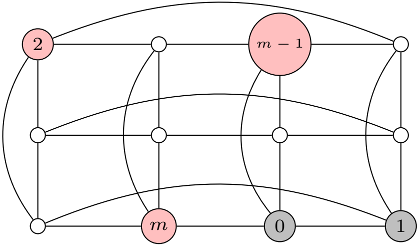

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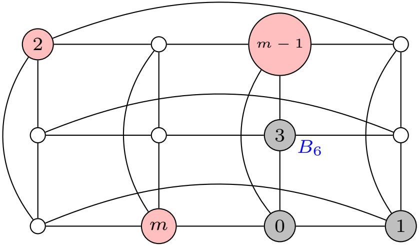

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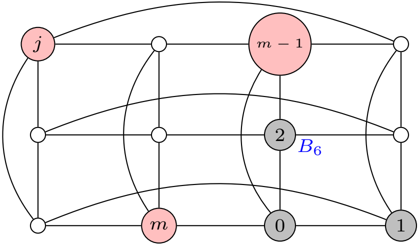
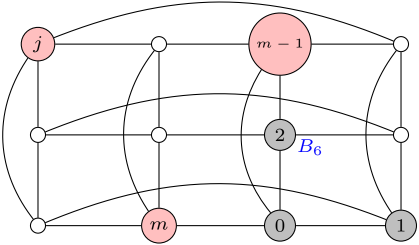


- If $j \neq 2$, then Bob assigns label 2 to the free neighbour of u.
- Bob cancels all Alice's possibilities of creating the edge label $m-2$.



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- The game ends if there is no legal move possible or a RRG labeling is created.
- Alice wins if a RRG labeling of G is created, otherwise Bob wins.

Question (Tuza, 2017)

Given a simple graph G and a set of consecutive nonnegative integer labels $\mathcal{L} = \{0, \ldots, k\}$, for which values of k can Alice win the range-relaxed graceful game?

Theorem (Oliveira, Dantas and Luiz, 2020)

Let G be a simple graph on n vertices and maximum degree Δ. Alice wins the range-relaxed graceful game on G for any set of integer labels $\mathcal{L} = \{0, \ldots, k\}$ with

$$k \geqslant (2\Delta^2 + 1)(n-1) + (2\Delta + 1)\binom{n-1}{2}.$$

Sketch of the Proof

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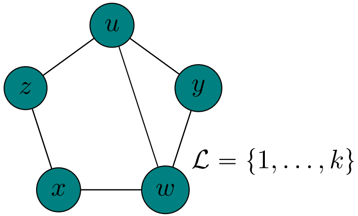
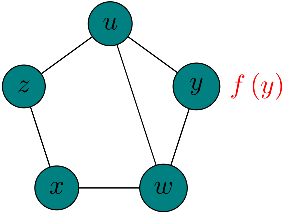


At the k-th move, a player assigns a label $f(v) \in \mathcal{L}_v, \forall v \in V(G)$. The set $\mathcal{L}_u, \forall u \in V(G)$ is updated according to the following four steps:

(Step 1) For every free vertex $u \in V(G)$, remove $f(v)$ from $\mathcal{L}_u$.



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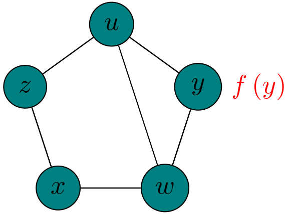
(Step 1) For every free vertex $u \in V(G)$, remove $f(v)$ from $\mathcal{L}_u$.

(Step 2) For every free vertex $u \in N(v)$, we delete from $\mathcal{L}_u$ every label l such that $| l - f(v) | = g(e), \forall e \in E(G)$ that has both endpoints labeled.



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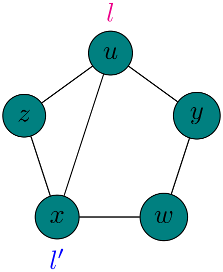
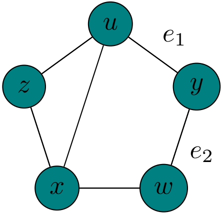
(Step 3) For every vertex $u \in N(v)$ and for every $w \in N(u) \cap N(v)$, such that w is already labeled and $f(w) \equiv f(v) \pmod 2$ delete from $\mathcal{L}_u$ the label $\frac{f(w)+f(v)}{2}$.



Parity condition: $\frac{l+l'}{2}$

(Step 4) For every free vertex $u \in V(G)$ and for every vertex $u' \in N(u)$, such that u' is already labeled, we delete from $\mathcal{L}_u$ every label l such that $\mid l - f(u') \mid = g(vv')$, for all $v' \in N(v)$.



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- We conclude that at most $(n-1) + 2\Delta\binom{n-1}{2} + \binom{n-1}{2} + 2\Delta^2\,(n-1)$ labels are deleted from each set of available labels.

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□

Conclusion

Cartesian Product	Winner
$P_r \square P_s; r, s \geqslant 2$	**B**
$B_{q,r}, q \geqslant 2, r \geqslant 1$	**B**
$P_q \square K_p, p, q \geqslant 2$	**B**
$T_{p,q}, p \equiv 0 \mod 4, q \equiv 0 \mod 2$	**B**
$Y_{m,n}, m \geqslant 3$	**B**

Corona and Other Classes	Winner
$C_n \odot K_1, n \geqslant 3$	**B**
Crown Graph $S_n^0, n > 3$	**B**
kC_n-snakes, $k \geqslant 2, n \geqslant 3$	**B**

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$C_n \odot K_1, n \geqslant 3$	B
$P_q \square K_p, p, q \geqslant 2$	B
$P_r \square P_s; r, s \geqslant 2$	B
$B_{q,r}, q \geqslant 2, r \geqslant 1$	B

Alice wins	Bob wins
C_3, K_1	$C_3 \odot K_1$
K_3, P_2	$K_3 \square P_2$
P_2, P_3	$P_2 \square P_3$
S_r, P_2	B_{2r}

Conclusion

Graph Class	Results of RRG Game
Simple graph on n vertices	$k \geqslant (2\Delta^2 + 1)(n-1) + (2\Delta + 1)\binom{n-1}{2}$
Trees	$k \geqslant (2\Delta^2 + 1)(n-1) + 2\Delta(n-2)$

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Thank you for your attention.