The monochromatic transversal game on clique-hypergraphs of powers of cycles

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Summary

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1 Preliminaries

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2 Playing the game

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3 Conclusions

Standard definitions

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- $G = (V, E)$ is an *undirected simple graph*.

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- *Clique*: subset S of V inducing a complete graph.
- *Maximal clique*: clique not properly contained in any other clique.

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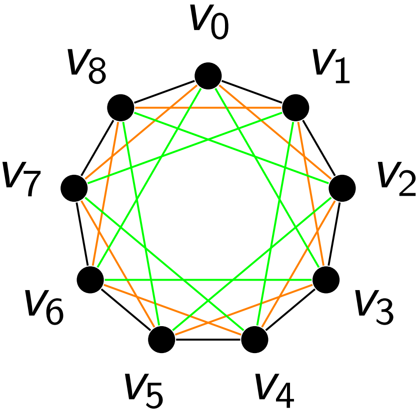
Example of maximal clique

$$G = C_9^3$$



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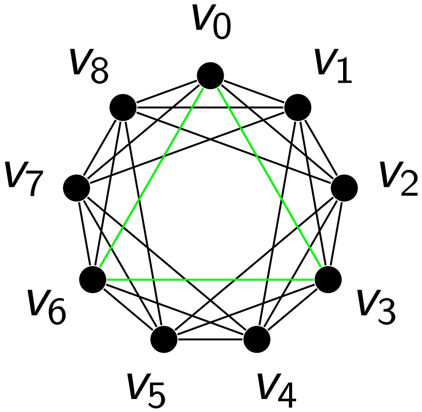
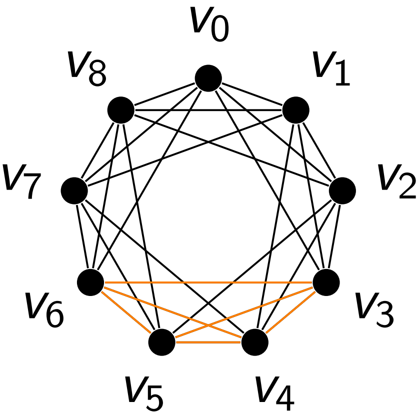
Example of maximal clique

$$S_1 = \{v_0, v_3, v_6\}$$



Example of maximal clique

$$S_2 = \{v_3, v_4, v_5, v_6\}$$



Hypergraphs

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Hypergraphs

- Denoted by $\mathcal{H} = (\mathcal{V}, \mathcal{E})$.

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- $\mathcal{V}$ is a finite set of vertices.
- $\mathcal{E}$ is a family of nonempty subsets of $\mathcal{V}$ called *hyperedges*.

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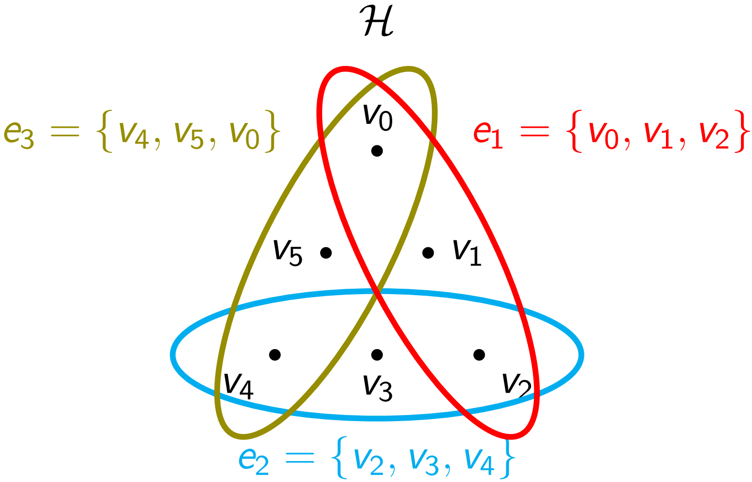
Example of hypergraph



Clique-hypergraphs

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Clique-hypergraphs

- Hypergraph $\mathcal{H}(G) = (V, \mathcal{E})$ induced from G.

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Clique-hypergraphs

- Hypergraph $\mathcal{H}(G) = (V, \mathcal{E})$ induced from G.
- V is the same of G.

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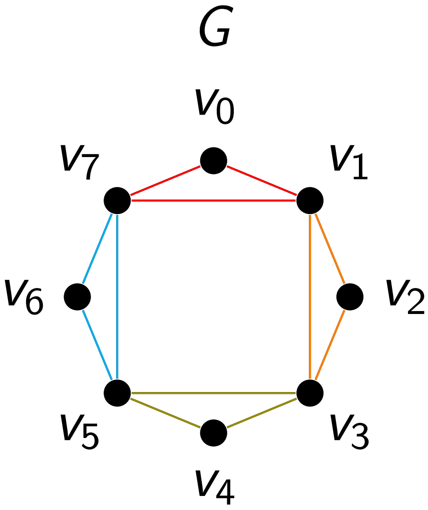
Clique-hypergraphs

- Hypergraph $\mathcal{H}(G) = (V, \mathcal{E})$ induced from G.
- V is the same of G.
- Hyperedges in $\mathcal{E}$ are the maximal cliques of G.

Example of clique-hypergraph



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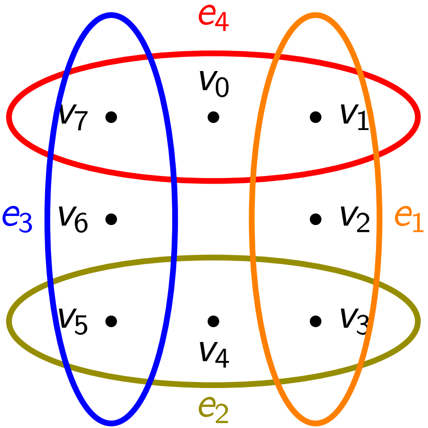
Example of clique-hypergraph

$\mathcal{H}(G)$



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Historical references

- 1991 Duffus et al.:

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  * Introduced clique-hypergraphs; and

Historical references

- 1991 Duffus et al.:
  - Introduced clique-hypergraphs; and
  - What is the *smallest number of colours* needed to colour each $v \in V(\mathcal{H}(G))$ s.t. no pair of adjacent vertices is monochromatic (*clique-chromatic number*)?

Historical references

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  - $\mathcal{H}(G)$ of any F-free graph G can be $f(F)$-coloured $\Leftrightarrow$ all components of F are paths.

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- 2013 Campos et al.:
  * answered Duffus' question in the case of C_n^k.
- 2014, Bacsó et al.:
  * clique-chromatic number is 3 for almost all perfect graphs.

Transversal

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Transversal

- Defined over a hypergraph $\mathcal{H}$.

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Transversal

- Defined over a hypergraph $\mathcal{H}$.
- Subset of vertices of V that has a nonempty intersection with every hyperedge of $\mathcal{H}$.

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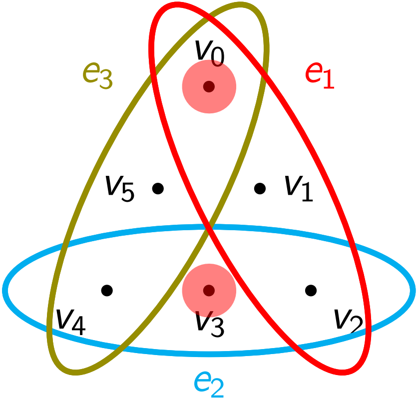
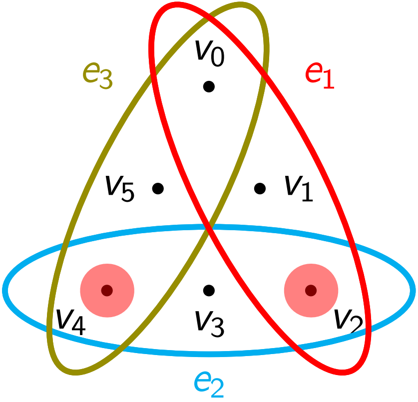
Example of transversal

$\{v_0, v_3\}$ is a transversal in $\mathcal{H}$



Example of transversal

$\{v_2, v_4\}$ is a transversal in $\mathcal{H}$



Combinatorial games

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Combinatorial games

- Alternating finite two-player game of pure strategy.

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- All relevant information is public.

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Combinatorial games

- Alternating finite two-player game of pure strategy.
- All relevant information is public.
- No luck or randomness are allowed.

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Historical references

- 2016∼2017 Bujtás et al.:

Historical references

- 2016∼2017 Bujtás et al.:
  - studied a combinatorial game using the concept of tranversals in hypergraphs, the *transversal game*:

Historical references

- Played on a hypergraph $\mathcal{H}$.

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Historical references

- Played on a hypergraph $\mathcal{H}$.
- Two players, Edge-hitter and Staller, alternately take turns choosing a vertex.

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Historical references

- Played on a hypergraph $\mathcal{H}$.
- Two players, Edge-hitter and Staller, alternately take turns choosing a vertex.
- Each chosen vertex must hit at least one edge not hit by the vertices previously chosen.

Historical references

- Game ends when the set of chosen vertices becomes a transversal of $\mathcal{H}$.

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- Edge-hitter tries to minimize the number of chosen vertices.

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Historical references

- Game ends when the set of chosen vertices becomes a transversal of $\mathcal{H}$.
- Edge-hitter tries to minimize the number of chosen vertices.
- Staller tries to maximize it.

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Monochromatic transversal game

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- Two players, Alice and Bob, alternately take turns.

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Monochromatic transversal game

- Played on a hypergraph $\mathcal{H}$.
- Two players, Alice and Bob, alternately take turns.
- Alice colours red a vertex.

Monochromatic transversal game

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- Alice colours red a vertex.
- Bob colours blue a vertex.

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Monochromatic transversal game

- Alice wins if there is red transversal.

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Monochromatic transversal game

- Alice wins if there is red transversal.
- Bob wins if there is a monochromatic blue hyperedge.

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Monochromatic transversal game

- Alice wins if there is red transversal.
- Bob wins if there is a monochromatic blue hyperedge.
- Both players are enabled to start the game and play optimally.

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Distance between two vertices in G

Distance between two vertices in G

- $d_G(u, v)$: number of edges in a shortest path in G from u to v.

k-th power of cycles

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k-th power of cycles

- Denoted by C_n^k.

k-th power of cycles

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- u and v are adjacent $\Leftrightarrow d_{C_n}(u, v) \leq k$ for $2 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

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Standard results on C_n^k

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- $C_n^1 \simeq C_n$ for $k = 1$.

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Standard results on C_n^k

- $C_n^1 \simeq C_n$ for $k = 1$.
- $C_n^k \simeq K_n$ for $k \geq \lfloor \frac{n}{2} \rfloor$.

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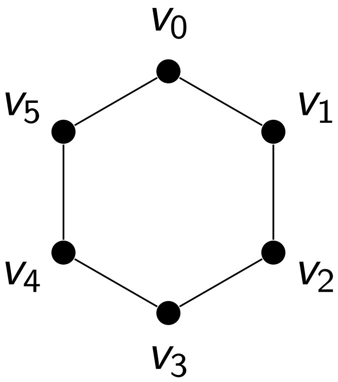
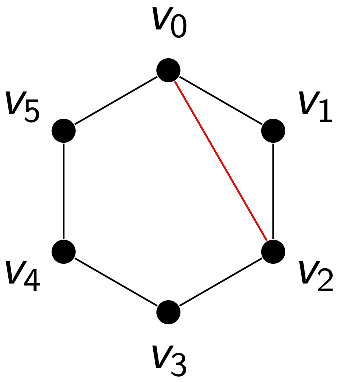
Example of k-th power of cycles

$$C_6 \; ; \; d_{C_6}(v_i, v_j) = 1, \;\; i \neq j$$



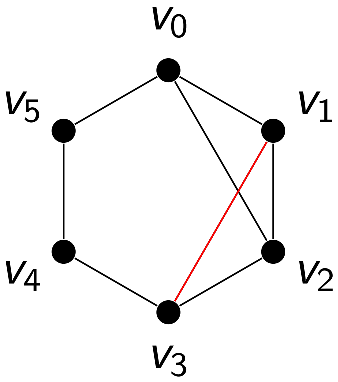
Example of k-th power of cycles

$$d_{C_6}(v_0, v_2) = 2$$



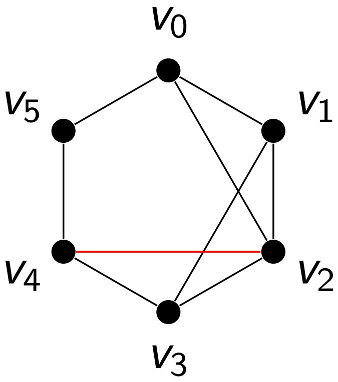
Example of k-th power of cycles

$$d_{C_6}(v_1, v_3) = 2$$



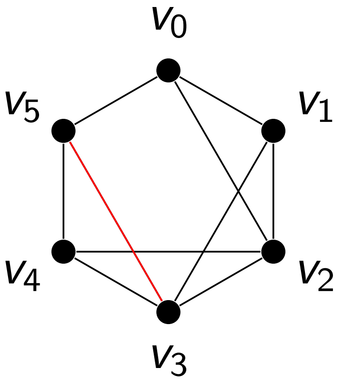
Example of k-th power of cycles

$$d_{C_6}(v_2, v_4) = 2$$



Example of k-th power of cycles

$$d_{C_6}(v_3, v_5) = 2$$



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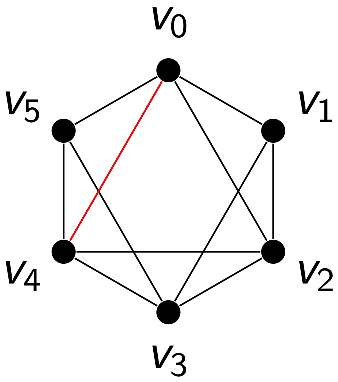
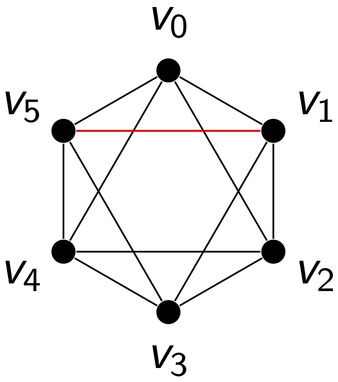
Example of k-th power of cycles

$$d_{C_6}(v_4, v_0) = 2$$

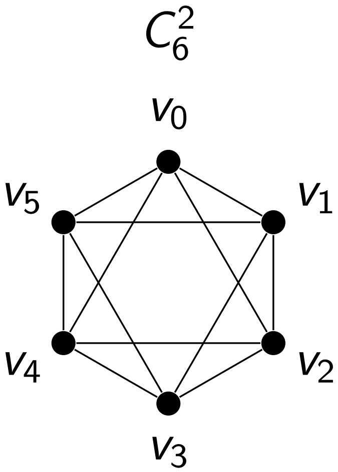


Example of k-th power of cycles

$$d_{C_6}(v_5, v_1) = 2$$



Example of k-th power of cycles



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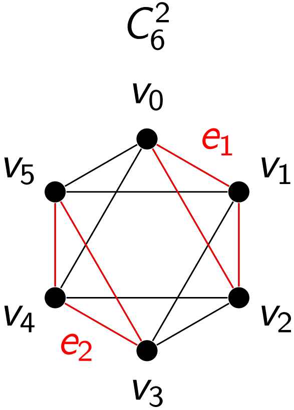
External maximal clique

- Set of $k+1$ vertices with consecutive indexes.



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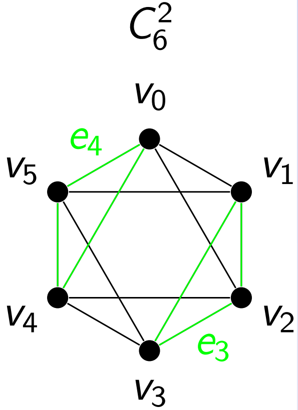

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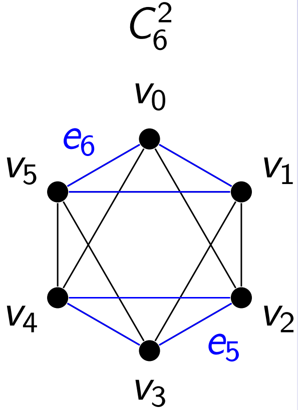
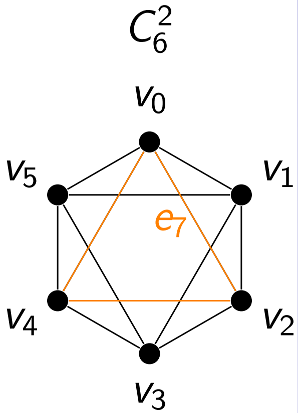


Internal maximal clique

- Vertex set with nonconsecutive indices inducing a maximal clique.



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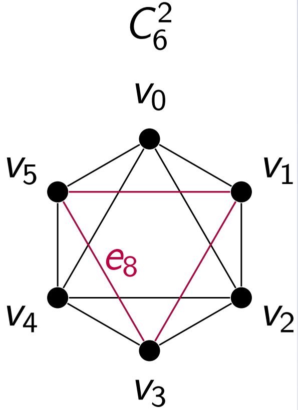

Playing the game

Remark 1

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Playing the game

Remark 1

If there exists a strategy that allows Alice (resp. Bob) to win when Bob (resp. Alice) starts the game, then there exists a strategy that allows Alice (resp. Bob) to win when she (resp. he) starts the game.

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The game over $\mathcal{H}(C_n^2)$

Theorem 2

Let $n > 6$. There exists a strategy that allows Alice to win the game played on the clique-hypergraph $\mathcal{H}(C_n^2)$, independently of who starts playing the game.

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The game over $\mathcal{H}(C_n^2)$

Proof

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The game over $\mathcal{H}(C_n^2)$

Proof

1° C_n^2 for $n > 6$ has only external maximal cliques that are all K_3.

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1° C_n^2 for $n > 6$ has only external maximal cliques that are all K_3.

2° Bob must colour blue 3 consecutive vertices to win.

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The game over $\mathcal{H}(C_n^2)$

Proof

1° C_n^2 for $n > 6$ has only external maximal cliques that are all K_3.

2° Bob must colour blue 3 consecutive vertices to win.

3° By Remark 1 it's sufficient make the proof when Bob starts playing.

The game over $\mathcal{H}(C_n^2)$

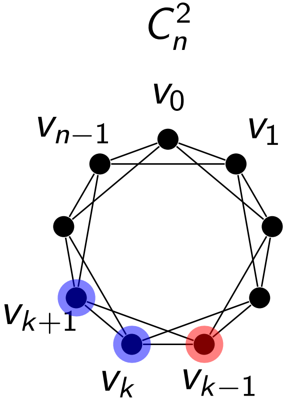
Proof

4° Alice's strategy is following the rules which are stated in decreasing importance (she only follows rule (j) if its not possible to follow any rule (i) for $i < j$):

The game over $\mathcal{H}(C_n^2)$

Proof

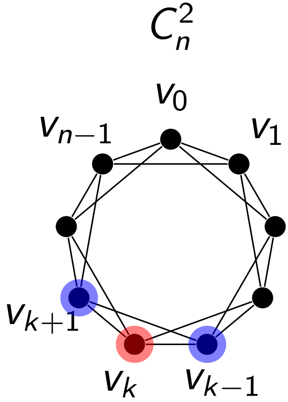
(i) she colours vertex $v_{k-1 \pmod n}$ whenever there are two blue vertices $v_k \pmod n$ and $v_{k+1} \pmod n$;



The game over $\mathcal{H}(C_n^2)$

Proof

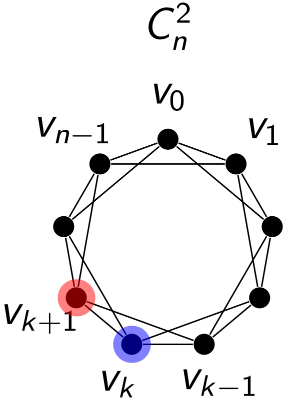
(ii) she colours vertex $v_k \pmod n$ whenever there are two blue vertices $v_{k-1} (\bmod\, n)$ and $v_{k+1} (\bmod\, n)$;



The game over $\mathcal{H}(C_n^2)$

Proof

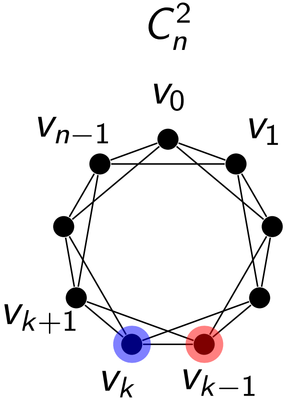
(iii) if Bob coloured vertex $v_k(\bmod n)$ then Alice colours vertex $v_{k+1}(\bmod n)$; and



The game over $\mathcal{H}(C_n^2)$

Proof

(iv) if Bob coloured vertex $v_k(\bmod n)$ then Alice colours vertex $v_{k-1}(\bmod n)$.



The game over $\mathcal{H}(C_n^k)$

Theorem 3

Let $n > 6$ and $k \geq 2$. If C_n^k has no internal maximal cliques, then there exists a strategy that allows Alice to win the game played on $\mathcal{H}(C_n^k)$, independently of who starts playing.

The game over $\mathcal{H}(C_n^k)$

- If Bob can't colour a clique of size k', then he can't colour a clique of size k for $k \geq k'$.

The game over $\mathcal{H}(C_n^k)$

Lemma 1

Let $n > 6$ and $2 \leq k' < k < \lfloor \frac{n}{2} \rfloor$. If Alice wins the game on $\mathcal{H}(C_n^{k'})$ and C_n^k do not have any internal maximal cliques, then Alice wins the game on $\mathcal{H}(C_n^k)$.

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The game over $\mathcal{H}(C_n^k)$

Proof (Theorem 3)

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The game over $\mathcal{H}(C_n^k)$

Proof (Theorem 3)

1° By Theorem 2 Alice always wins the game on $\mathcal{H}(C_n^2)$ for $n > 6$.

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The game over $\mathcal{H}(C_n^k)$

Proof (Theorem 3)

1° By Theorem 2 Alice always wins the game on $\mathcal{H}(C_n^2)$ for $n > 6$.

2° By the hypotheses C_n^k has no internal maximal cliques.

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The game over $\mathcal{H}(C_n^k)$

Proof (Theorem 3)

1° By Theorem 2 Alice always wins the game on $\mathcal{H}(C_n^2)$ for $n > 6$.

2° By the hypotheses C_n^k has no internal maximal cliques.

3° By Lemma 1 Alice always wins the game played on $\mathcal{H}(C_n^k)$.

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Conclusions

We summarize our results for the monochromatic transversal game in the following table.

	Value for n	Value for k	Alice	Bob
K_n	$n \geq 2$	1	A	A
P_n	$3 \leq n \leq 5$	1	A	B
P_n	$n \geq 6$	1	B	B
C_n	$n \geq 4$	1	B	B
C_n^k	6	2	B	B
C_n^k	$n > 6$	$2 \leq k \leq \lfloor \frac{n}{2} \rfloor$	A	A

Table 1: Entries A = Alice wins when starts; B = Bob wins when starts.

Thank you for your attention.

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