Fair allocation of indivisible goods under conflict constraints

Nina Chiarelli, Matjaž Krnc, Martin Milanič, Nevena Pivač
University of Primorska, Koper, Slovenia

Ulrich Pferschy and Joachim Schauer
University of Graz, Austria

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Introduction of the Problem

Fair Division of Indivisible Goods

Given: A set V of items, k agents with profit functions $p_1, \ldots, p_k : V \rightarrow \mathbb{Z}_+$
Task: Compute a k-partition of V with maximum satisfaction level

let $V_1, V_2, \ldots, V_k$ be the partition of the items and $P_i = \sum_{v \in V_i} p_i(v)$ the profit of agent i

Satisfaction Level: $\min_i P_i$ profit of the least happy agent

Also called Santa Claus Problem: you have k kids and n presents and want to distribute them in a fair way.

Make the least happy kid as happy as possible!

Introduction

- Bezakova and Dani 2005: approximation guarantee of $(2-\varepsilon)$ impossible (under $\mathcal{P} \neq \mathcal{NP}$)
- Bansal and Sviridenko 2006 coined the name Santa Claus Problem, $O\left(\frac{\log\log k}{\log\log\log k}\right)$–approximation for a restricted version where $p_i(v) \in \{0, p(v)\}\ \forall v$.
- since then many more results and variations, most of these variants remain hard

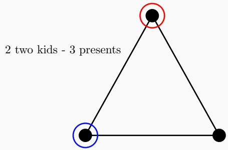
Our Motivation

- *practical case:* only a constant number of kids
- A conflict graph additionally gives a structure on the items. Certain pairs of items should not be assigned to the same agent.
  - my daughter does not want two remote controlled cars for christmas
  - my son wants skis or a snowboard, but cannot use both in his skiing course
- items correspond to vertices in the conflict graph
- edges connect pairs of incompatible items/presents

Our Motivation

Observation

- A feasible allocation must be an independent set for each agent.
- A partition of the items may well be infeasible.
  $\implies$ find instead a partial k coloring.



Results

Observation

Hard, inapproximable problem even for one kid:

Fair 1-division under conflicts $\Longleftrightarrow$ Independent Set Problem

Our strategy: look at special conflict graphs!

Fair 2-Division without conflicts

This problem is already polynomial equivalent to the binary knapsack problem and thus weakly $\mathcal{NP}$-hard.

The best we can hope for are pseudopolynomial algorithms!

Results

Negative Results

- **Bipartite Graphs**

  For each integer $k \geq 2$, Fair k-Division Under Conflicts is strongly $\mathcal{NP}$-hard in the class of bipartite graphs.
- Other structural results to extend inapproximability results

Results

Positive Results, Tractable Graph Classes (pseudopolynomial)

- cocomparability
- chordal
- bounded clique-width ($\Longrightarrow$ bounded treewidth)
- biconvex bipartite graphs

Main Feature of all Pseudopolynomial Algorithms

Introduce k-tuples $(p_1, \ldots, p_k)$ for all possible profit combinations.

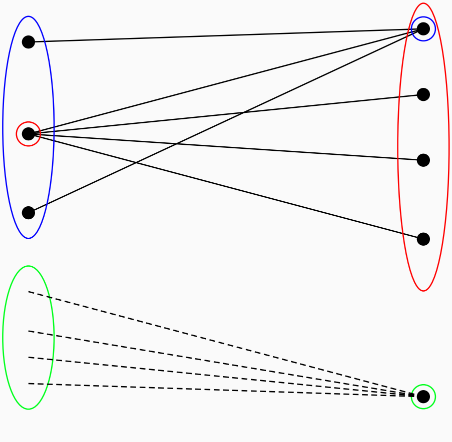
The number of these tuples is polynomial, since k is constant and the profit coefficients can grow only polynomial in the input.

Goal: Find all reachable feasible profit profiles.

- Start with a small graph, for which you can decide for any profit tuple if it is feasible.
- Add the remaining vertices and edges and complement the feasible tuples.

Biconvex Bipartite Graphs

A bipartite graph $G = (A \cup B, E)$ is biconvex if there is an ordering of A and B that fulfills the adjacency property,
i.e. for every vertex $a \in A$ the neighborhood $N(a)$ is a consecutive interval in B and vice-versa.



Biconvex Graphs

Structure theorem (Abbas, Stewart [2000])

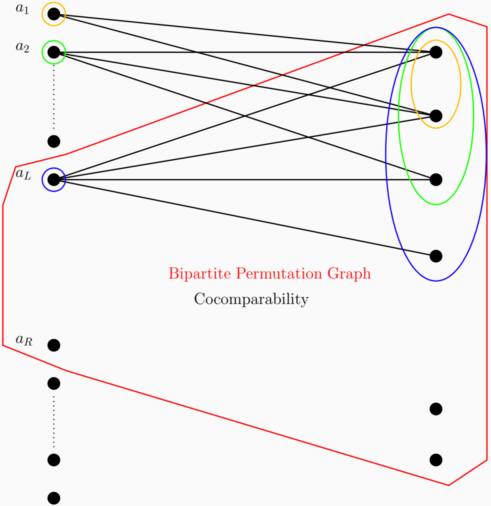
Let $G = (A \cup B, E)$ be a connected biconvex bipartite graph. Then there exists a biconvex ordering of the vertices of G such that:

(i) For all a_i, a_j with $a_1 \leq a_i < a_j \leq a_L$ or $a_R \leq a_j < a_i \leq a_p$ there is $N(a_i) \subseteq N(a_j)$.

(ii) The graph G' induced by vertex set $\{a_L, \dots, a_R\} \cup B$ is a connected bipartite permutation graph.

$\Longrightarrow$ cocomparability graph, pseudopolynomial solvable!

Biconvex Graphs



Biconvex Graphs

Observation:

Whenever a vertex a_l is chosen for a kid j, all vertices a_i with $i \leq l$ are also feasible.

- decide for any kid j, the highest index of a vertex from $a_1, a_2, \ldots, a_{L-1}$ assigned to that kid ("guess" all possibilities)
- for all adjacent vertices in B: set all profits in the bipartite permutation graph to 0
- now use the cocomparability result to solve the bipartite permutation part (i.e. fill the dynamic programming profile)
- augment the dynamic programming profile by all vertices from $a_1, a_2, \ldots, a_{L-1}$ with index lower than the highest index for the respective kid.

Biconvex Graphs

Running time:
All together, there are $O(n^{2k})$ "guesses" and a running of $O(n^{k+2}Q^k)$ for each subproblem on a permutation graph.
(Q is the highest reachable profit over all kids)

$\Longrightarrow$ pseudopolynomial

Last Slide

Santa Claus in Ischia has a really tough job!

Writing your wish list without conflicts would help him a lot!

Thank you for your attention!