Hypergraph aspect in equitable colorings of some block graphs

joint work with Vahan Mkrtchyan and Janusz Dybizbański

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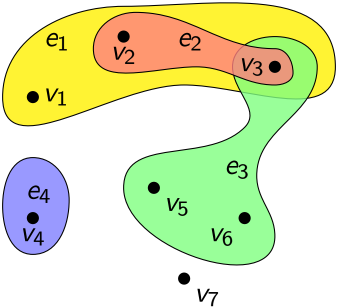
A *hypergraph* $\mathcal{H}$ is a pair $(V, \mathbb{E})$, where V is the n element set of vertices of $\mathcal{H}$ and $\mathbb{E}$ is a family of m non-empty subsets of V called edges or hyperedges.



$$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\};\ n = 7$$
$$\mathbb{E} = \{\{v_1, v_2, v_3\}, \{v_2, v_3\}, \{v_3, v_5, v_6\}, \{v_4\}\};\ m = 4$$

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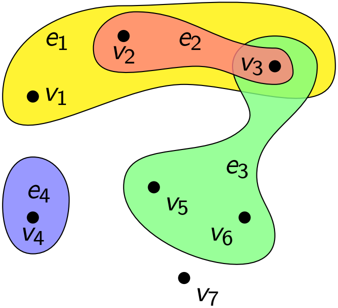

We are interested in *some kind* of edge coloring of hypertrees.

Host graph

We say that a hypergraph $\mathcal{H}$ has an *underlying* (*host*) graph G (spanned on the same set of vertices) if each hyperedge of $\mathcal{H}$ induces a connected subgraph in G. Furthermore, it is assumed that for each edge e_G in G there exists a hyperedge $e_{\mathcal{H}}$ in $\mathcal{H}$ such that $e_G \subseteq e_{\mathcal{H}}$.

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Hypertree

Hypertree is defined as a hypergraph that there exists its underlying graph being a tree.

Definition

The *line graph* $L(\mathcal{H})$ of hypergraph $\mathcal{H}$ is a simple graph representing adjacencies between hyperedges in $\mathcal{H}$. More precisely, each hyperedge of $\mathcal{H}$ is assigned a vertex in $L(\mathcal{H})$ and two vertices in $L(\mathcal{H})$ are adjacent if and only if their corresponding hyperedges share a vertex in $\mathcal{H}$.

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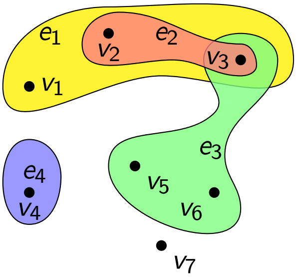
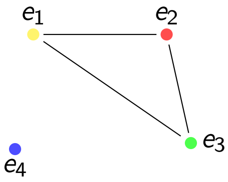


$\mathcal{H}$



$L(\mathcal{H})$

Some observations

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- A graph G is chordal if and only if it is a line graph of a hypertree [Duchet, 1978] (chordal = all cycles of four or more vertices have a chord)

Corollary

An edge coloring of hypertrees is equivalent to vertex coloring of chordal graphs.

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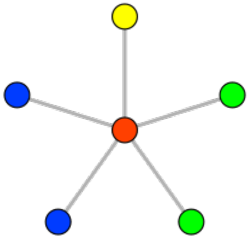

$\chi_=(K_{1,5}) = 4$

Observation

For a general graph G if it admits an equitable vertex t-coloring it does not imply that it admits an equitable vertex $(t+1)$-coloring (cf. for example $t = 2$ and $G = K_{3,3}$).

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Equitable chromatic spectrum - the set of colors admitting equitable coloring of the graph.
If $\chi^*_=(G) = \chi_=(G)$ then we say that the equitable chromatic spectrum of G is *gap-free*.

Equitable vertex coloring of chordal graphs

Proposition [Bodlaender et al.]

The problem of equitable vertex coloring of interval graphs is NP-hard.

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Some known results

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Corollary

The problem of an equitable k-coloring is solvable in polynomial time for chordal graphs with bounded maximum clique size.

Block graphs - subclass of chordal graphs

Definition

Graph is a *block graph* (clique tree) if every block (maximal 2-connected component) is a clique.

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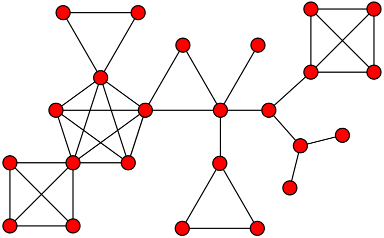

Parametrized complexity

Remark

For block graphs, it was shown [Gomes et al. 2019] that the problem is W[1]-hard with respect to the treewidth, diameter and the number of colors. This in particular means that under the standard assumption FPT$\neq$W[1] in parameterized complexity theory, the problem is not likely to be polynomial time solvable in block graphs.

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Some other parameters ...

... are the subject of our present project ...

Lower bound

G - arbitrary graph

$$\chi_=(G) \geq \max\left\{\omega(G), \left\lceil \frac{|V(G)|+1}{\alpha_{min}(G)+1} \right\rceil\right\}$$

$\alpha_{\mathrm{min}}(G) = \min_{v \in V(G)} \alpha(G, v)$
$\alpha(G, v)$ - the size of the largest independent set of G containing v

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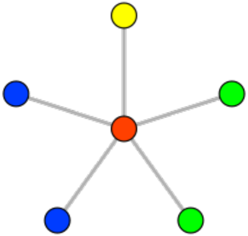

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Conjecture 1

For any block graph G, we have:

$$\chi_=(G) \leq 1 + \max\left\{\omega(G), \left\lceil \frac{|V(G)|+1}{\alpha_{min}(G)+1} \right\rceil\right\}$$

Well-covered block graphs

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Well-covered graphs - graphs fulfilling the condition

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$$\frac{|V(G)|+1}{\alpha_{min}(G)+1} \leq \frac{\alpha(G)\cdot\omega(G)+1}{\alpha(G)+1} \leq \omega(G).$$

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Conjecture for well-covered block graphs

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There is one basic class of such graphs, namely complete graphs. Now, we define the following operation: let H be a block graph and let v be a vertex of H. Add a clique Q of size at least 2, $|Q| \geq 2$, to H ($Q \cap H = \{v\}$), and add one pendant clique to each vertex of Q except v.

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- recursive characterization of well-covered block graphs
- we show that every equitable k-coloring of H can be extended into the graph H with added clique Q with their pendant cliques - the algorithm is based on modified Ferrers matrices (details: https://arxiv.org/abs/2002.10151)

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Corollary

The equitable chromatic spectrum of well-covered block graphs is gap-free.

Block-graphs with small $\alpha_{\min}$

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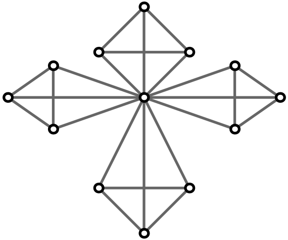

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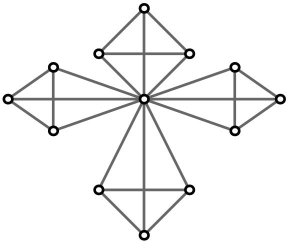

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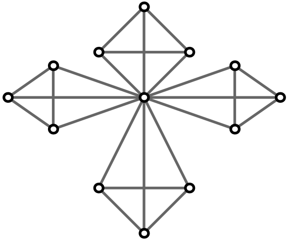

We need to partition the vertex set of G into minimum number of color classes of size at most 2.
This is equivalent to finding a maximum matching in $\overline{G - v}$, i.e in a complete multipartite graph.

$\alpha_{\min}(G) = 2$

Theorem

Let G be a connected block-graph with $\alpha_{\min}(G) = 2$. Then G is equitably k-colorable for every $k \geq \max\left\{\omega(G), \left\lceil \frac{|V(G)|+1}{\alpha_{min}(G)+1} \right\rceil\right\}$.

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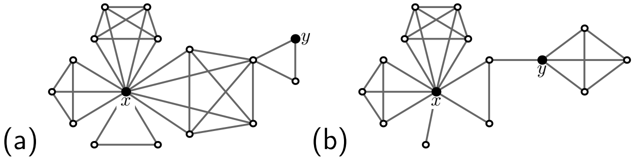
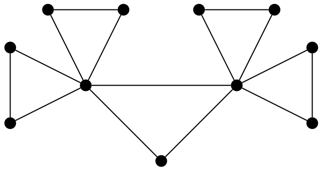


An example of two connected block graphs with $\alpha_{\min} = 2$.

$\alpha_{\min}(G) = 3$



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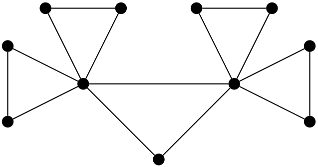

$|V| = 11$, $\omega(G) = 3$ and the cut-vertices attain the $\alpha(G, v) = 3$.

Thus, $\max\left\{\omega(G), \left\lceil \frac{|V(G)|+1}{\alpha_{min}(G)+1} \right\rceil\right\} = 3$.

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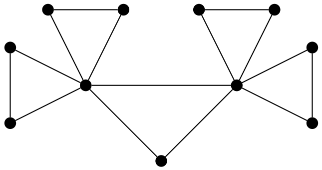

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One can check that G does not admit an equitable 3-coloring, and G is equitably 4-colorable. Hence $\chi_{=}(G) = 4 = 1 + \max\{\}$.

Future work ...

- To prove Vizing-Goldberg type conjecture for all block graphs.

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Conjecture

For any block graph G, we have:

$$\max\left\{\omega(G), \left\lceil \frac{|V(G)|+1}{\alpha_{min}(G)+1} \right\rceil\right\} \leq \chi_=(G) \leq$$

$$\leq 1+\max\left\{\omega(G), \left\lceil \frac{|V(G)|+1}{\alpha_{min}(G)+1} \right\rceil\right\}.$$

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Future work ...

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- For which graph classes right inequality from Conjecture 1 holds?
  Remark 1: Conjecture 1 does not hold for all graph classes! (cf. $K_{3,5,7}$).
  Remark 2: Conjecture 1 does not hold for all chordal graphs!

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THANK YOU FOR YOUR ATTENTION!