1,2,3 conjecture on some subclasses of bipartite graphs

CTW Extended Abstract

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Vertex coloring edge weighting

- An **edge k-weighting** is a function $\omega : E(G) \to [k] := \{1, 2, ..., k\}$.
- Label a vertex u by the sum of the weights of all edges incident to it, i.e. $\sum_{e \sim u} \omega(e)$.
- An edge k- weighting ω is a **proper vertex coloring** by sums if $\sum_{e \sim u} \omega(e) \neq \sum_{e' \sim v} \omega(e')$ for every $uv \in E(G)$.
- Denote by $\chi^e_\Sigma(G)$ the smallest value of k such that a graph G has an edge k-weighting which is a proper vertex coloring by sums, i.e., **vertex coloring k-edge weighting**.

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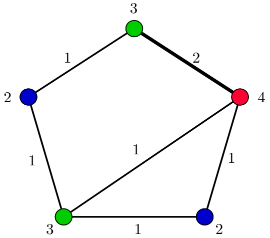
Example of vertex coloring 2-edge weighting of a graph



Vertex coloring 2-edge weighting of G

1-2-3 Conjecture

- In 2004, Karoński, Łuczak and Thomason[1] gave the 1-2-3 conjecture.

1-2-3 Conjecture

If G has no connected component isomorphic to K_2, then $\chi^e_\Sigma(G) \leq 3$.

- The best known bound for $\chi^e_\Sigma(G)$ where G is not isomorphic to K_2 has been established as $\chi^e_\Sigma \leq 5$ by Kalkowski et al.[2].

[1] Edge weights and vertex colours, Karoński, Michał and Łuczak, Tomasz and Thomason, Andrew, Journal of Combinatorial Theory, Series B, 2004

[2] Maciej Kalkowski, Michał Karonski, and Florian Pfender. Vertex-coloring edge-weightings: towards the 1-2-3-conjecture. Journal of Combinatorial Theory, Series B, 2010

1-2-3 Conjecture: Known Results

- Dudek and Wajc showed that determining whether a given graph has **vertex coloring 2-edge weighting is NP-complete** [3]. Lu, Yu and Zhang proved that **every 3-connected bipartite graphs admits vertex coloring 2-edge weighting** [4].
- Thomassen et al. recently established that determining $\chi^e_\Sigma(G)$ for a bipartite graph G can be done in polynomial time.[5]
- Baudon et al.[6] considered the minimum number of sums in vertex coloring edge weighting and obtained NP-complete results for balanced bipartite graphs.

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[3]Andrzej Dudek and David Wajc. On the complexity of vertex-coloring edge-weightings. Discrete Mathematics and Theoretical Computer Science, 2011

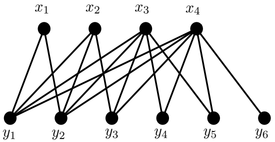
[4]Hongliang Lu, Qinglin Yu, and Cun-Quan Zhang. Vertex-coloring 2-edge-weighting of graphs. European Journal of Combinatorics, 2011

[5]Carsten Thomassen, Yezhou Wu, and Cun-Quan Zhang. The 3-flow conjecture, factors modulo k, and the 1-2-3-conjecture. Journal of Combinatorial Theory, Series B, 2016.

[6]Olivier Baudon, Julien Bensmail, Hervé Hocquard, Edge weights and vertex colours: Minimizing sum count Discrete Applied Mathematics 2019

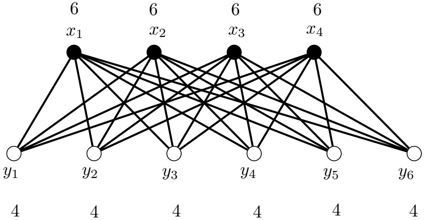
Result 1 : Vertex coloring 2-edge weighting of chain graphs

- A bipartite graph $G = (X, Y, E)$ with $|X| = p$ and $|Y| = q$, is called a *chain graph* if the neighborhoods of the vertices of X form a chain, i.e., the vertices of X can be linearly ordered, say $x_1, x_2, \ldots, x_p$ such that $N(x_1) \subseteq N(x_2) \subseteq \ldots \subseteq N(x_p)$.
- If $G = (X, Y, E)$ is a chain graph, then the neighborhoods of the vertices of Y also form a chain.
- An ordering $\sigma = (x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_q)$ of $X \cup Y$ is called a chain ordering if $N(x_1) \subseteq N(x_2) \subseteq \ldots \subseteq N(x_p)$ and $N(y_1) \supseteq N(y_2) \supseteq \ldots \supseteq N(y_q)$.
- It is known that every chain graph admits a chain ordering that can be computed in linear time.



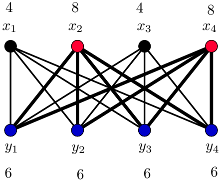
Complete bipartite graph

- Note that complete bipartite graphs form a proper subclass of chain graphs.
- The following vertex coloring $\{1, 2\}$-edge weighting, ω of a complete bipartite graph $G = (X \cup Y, E)$ can be computed in linear time using at most 3 colors.
- If $|X| \neq |Y|$ then $\omega(x_i y_j) = 1$ for each $x_i \in X, y_j \in Y$.

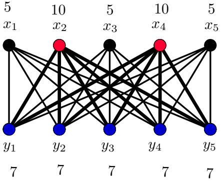


Otherwise, suppose $|X| = |Y| = r$. Then,

$$\omega(x_i y_j) = \begin{cases} 2, \text{ if } (i \bmod 2 = 0) \\ 1, \text{ otherwise, for each } 1 \leq i, j \leq r. \end{cases}$$



$r = 4$



$r = 5$

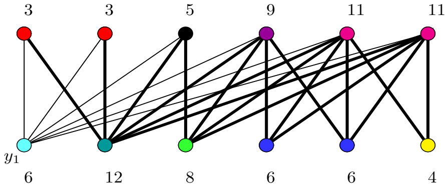
Note that this edge weighting induces a proper vertex coloring by sums since each vertex of X will be assigned color either r or $2r$ and the vertices of Y receive color $r + \left\lfloor \frac{r}{2} \right\rfloor$, where $\lfloor . \rfloor$ represents the greatest integer function.

Chain Graph

Let $G = (X, Y, E)$ be a chain graph with chain ordering, $\sigma = (x_1, x_2, \ldots, x_p, y_1, y_2, \ldots, y_q)$ of $X \cup Y$ such that $N(x_1) \subseteq N(x_2) \subseteq \ldots \subseteq N(x_p)$ and $N(y_1) \supseteq N(y_2) \supseteq \ldots \supseteq N(y_q)$.

Case 1 : $|X| = |Y| = 2r$

$$\omega(x_i y_j) = \begin{cases} 1, \text{ if } j = 1 \text{ and } 1 \leq i \leq 2r, \\ 2, \text{ otherwise.} \end{cases}$$

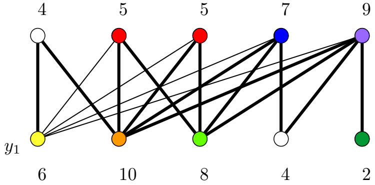


Since y_1 is adjacent to all $2r$ vertices of X and contribute 1 to each vertex of X, the vertices of Y receive even colors and the vertices of X receive odd colors.

Case 2 : $|X| = |Y| = 2r + 1$

Subcase (a)*:* $d(x_1) \leq r$

$$\omega_1(x_i y_j) = \begin{cases} 1, \text{ if } j = 1 \text{ and } 2 \leq i \leq 2r+1, \\ 2, \text{ otherwise.} \end{cases}$$

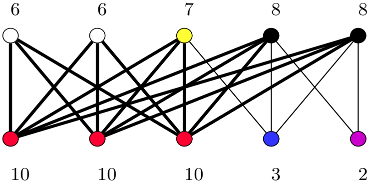


The vertices of Y receive even colors and the vertices of $X - \{x_1\}$ receive odd colors. Note that x_1 always receives an even color but since $d(x_1) \leq r$, x_1 may only be adjacent to vertices of Y that have the same parity but more value than x_1.

Subcase (b)*:* $d(x_1) \geq r+1$

Suppose $d(x_1) = r+1$. This implies that $\{y_1, \ldots, y_{r+1}\}$ are adjacent to each vertex of X. Consider the edge weights given by w_2:

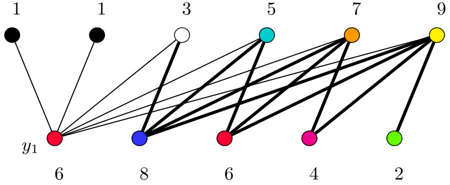
$$\omega_2(x_i y_j) = \begin{cases} 2, \text{ if } 1 \leq j \leq (r+1) \text{ and } 1 \leq i \leq 2r+1, \\ 1, \text{ otherwise.} \end{cases}$$



Note that $\{y_1, \ldots, y_{r+1}\}$ receive color $2(2r+1)$ and $\{y_{r+2}, \ldots, y_{2r+1}\}$ receive color at most $2r+1$. Since the vertices of X receive color at least $2(r+1)$, ω_2 is a proper vertex coloring by sums. A similar argument holds for $d(x_1) > r+1$.

Case 3 : If the number of vertices in one of the sets, X or Y, is even.
Suppose $|X| = 2l$, for some positive integer l. Since y_1 is adjacent to all $2l$ vertices of X, the following edge weighting given by ω result in odd colors for the vertices of X and even colors for the vertices of Y:

$$\omega(x_i y_j) = \begin{cases} 1, \text{ if } j = 1 \text{ and } 1 \leq i \leq 2l, \\ 2, \text{ otherwise.} \end{cases}$$

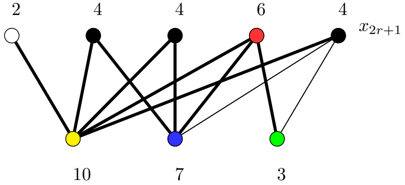


Case 4 : If $|X| = 2r+1$ and $|Y| = 2l+1$, $(l \neq r)$.

Let us assume, without loss of generality, that $r > l$.

Consider the edge weights given by ω:

$$\omega(x_i y_j) = \begin{cases} 1, \text{ if } i = 2r+1 \text{ and } 2 \leq j \leq 2l+1, \\ 2, \text{ otherwise.} \end{cases}$$



Note that all the vertices of Y except y_1 receive an odd color and each vertex of X receive an even color. Now any vertex of X receives color at most $2(2l+1)$ which is strictly less than $2(2r+1)$, the color of y_1 as $r > l$.

Theorem

If G is a chain graph, then the vertex coloring $\{1,2\}$-edge weighting of G can be computed in linear time. Thus, $\chi^e_\Sigma(G) = 2$.

Cartesian product of two graphs

The Cartesian product of two graphs G and H is the graph $G\Box H$ with vertices $V(G\Box H) = V(G) \times V(H)$, and for which $(x,u)(y,v)$ is an edge if $x = y$ and $uv \in E(H)$, or $xy \in E(G)$ and $u = v$.

Let G and H be graphs that have vertex coloring $\{1,2\}$-edge weighting, then $G\Box H$ has vertex coloring $\{1,2\}$-edge weighting that can be computed in linear time.

We define ω, vertex coloring $\{1,2\}$-edge weighting of $G\Box H$ for each edge $(x,u)(y,v)$ as follows: $\omega((x,u)(y,v)) = \begin{cases} \omega_H(uv), \text{ if } x = y \text{ and } uv \in E(H), \\ \omega_G(xy), \text{ if } xy \in E(G) \text{ and } u = v. \end{cases}$

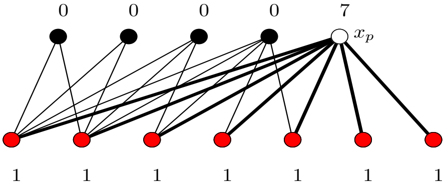
Suppose $x = y$ and $uv \in E(H)$. Since $x = y$, the total weights added to (x,u) and (y,u) due to the edges in G are the same.

Vertex coloring $\{0,1\}$-edge weighting

If G is a chain graph, then the vertex coloring $\{0,1\}$-edge weighting of G can be computed in linear time using 3 colors.

Without loss of generality, we assume that $|Y| \geq |X|$ and G is not a star graph. Consider the edge weights given by ω:

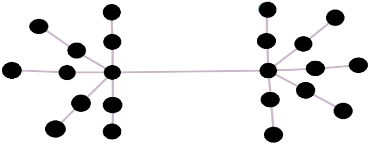
$$\omega(x_i y_j) = \begin{cases} 1, \text{ if } i = p \text{ and } 1 \leq j \leq q, \\ 0, \text{ otherwise.} \end{cases}$$



Note that all the vertices of Y are adjacent to x_p and ω assigns color 1 to each vertex of Y. Each vertex of X except x_p receive color 0 and x_p receives color $|Y|$.

Some subclasses of bipartite graphs that do not admit vertex coloring $\{0, 1\}$-edge weighting

1. C_{4r+2} and P_{4r+2}, $r \geq 1$.
2. Graphs formed by adding r distinct P_3 to each vertex of K_2.
3. Graphs formed by adding C_{4r+2} to each end vertex of P_{4r+3}.



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Thank You.