Polynomially solvable special cases of the Path-TSP

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joint work with

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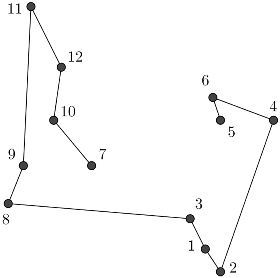

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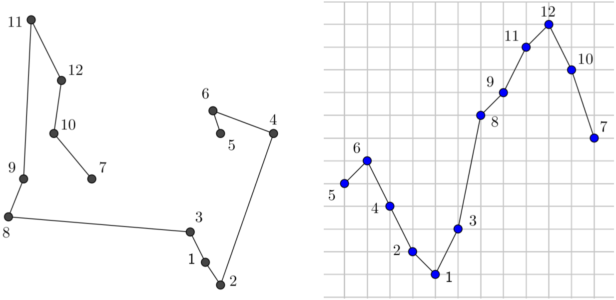

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Recently: the approximability of Path-TSP is the same as for TSP, up to an arbitrarily small error. (Traub et al. 2019)

Introduction: Polynomially solvable special cases (p.s.s.c.)

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The particular properties of the input are often represented in terms of inequalities fulfilled by the entries of the distance matrix C
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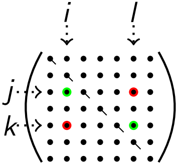

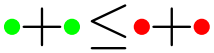

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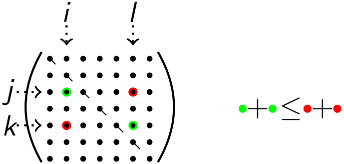

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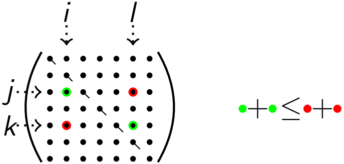

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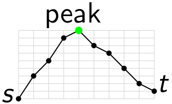

The Path-TSP with a Demidenko distance matrix

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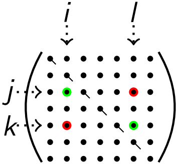

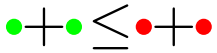

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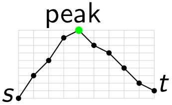

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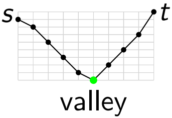

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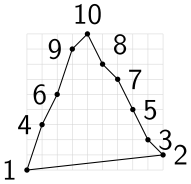

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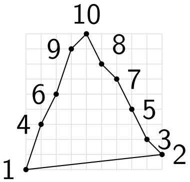

Theorem (Demidenko, 1979)
For a set of cities with a Demidenko distance matrix there exists an optimal tour which is pyramidal. In this case the TSP on n cities can be solved in $O(n^2)$ time by dynamical programming.

Forbidden pairs of arcs

In an (s,t)-path τ a pair of arcs $(i,\tau(i))$ and $(j,\tau(j))$ is called a forbidden pair of arcs if either $i<j<\tau(i)<\tau(j)$ or $i>j>\tau(i)>\tau(j)$ holds.

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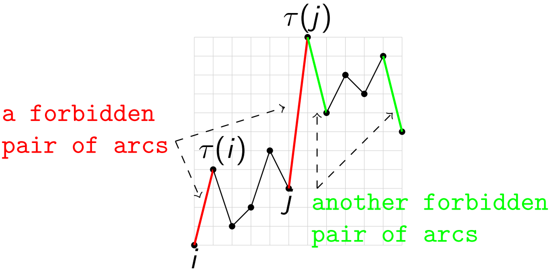

The forbidden pairs lemma: In the case of a Demidenko distance matrix there always exist an optimal (s,t)-TSP-path without forbidden pairs of arcs, for any s and t.

Proof of the forbidden pairs lemma

Exchange argument: replace the forbidden pair of arcs $(i, \tau(i))$, $(j, \tau(j))$ in an optimal path by the non-forbidden pair (i, j), $(\tau(i), \tau(j))$ and reverse the $(\tau(i), j)$-subpath. The resulting path is again optimal.

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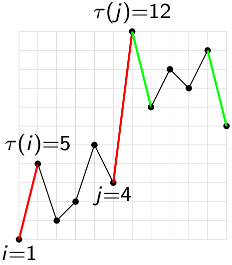

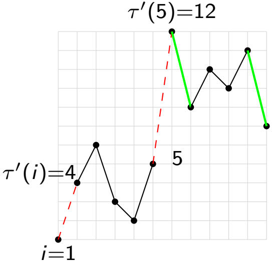

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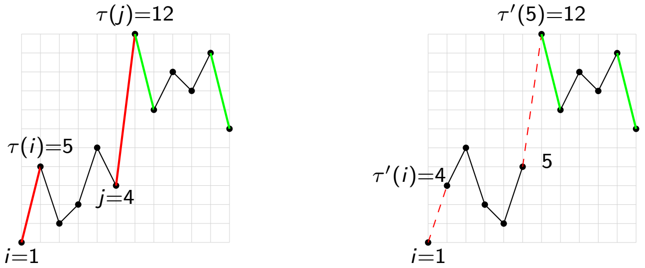

After repeating the exchange argument a finite number of times we get an optimal (s, t)-TSP-path without forbidden pairs of arcs.

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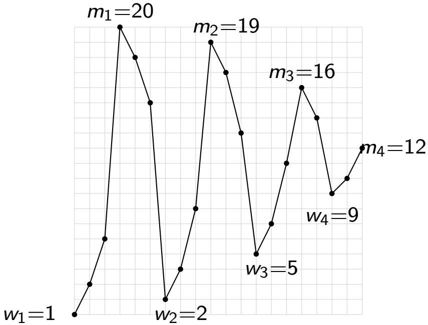
Lemma 2: For any t there exist an optimal $(1, t)$-TSP-path without forbidden pairs of arcs in which the peaks decrease and the valleys increase along the path.



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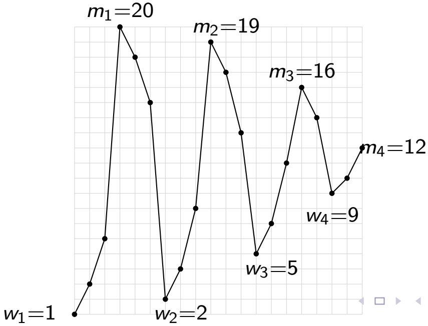

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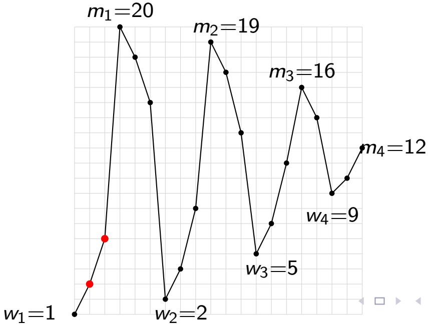

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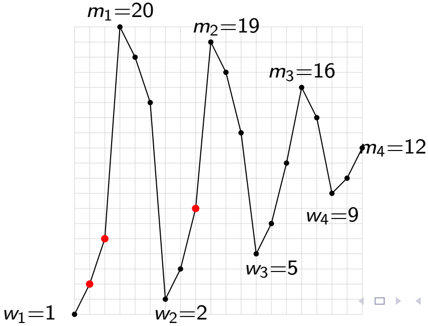

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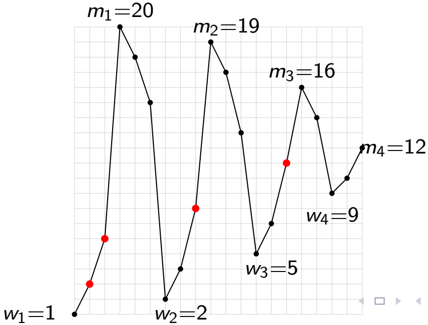

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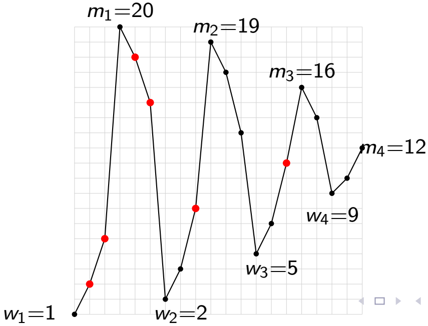

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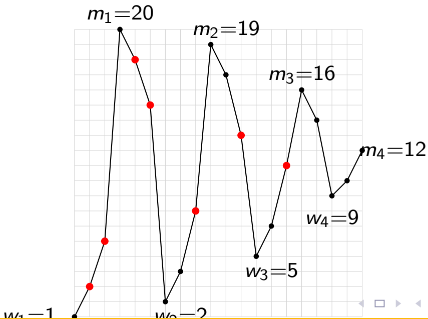

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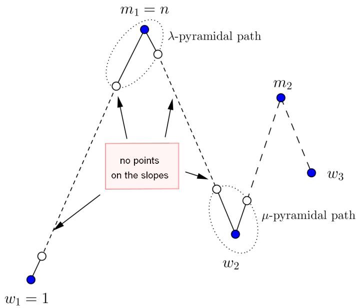

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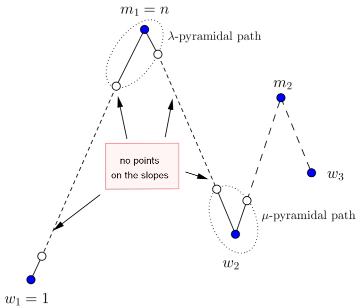

Theorem: For every destination city t the $(1, t)$-Path-TSP on n cities with a Demidenko distance matrix can be solved in $O(n^5)$ time by dynamic programming.

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(B) $i < j$ holds for each city i in the $(s,1)$-subpath and for each city j in the (n,t)-subpath,

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For $1 < s < t < n$ there is an optimal (s,t)-TSP-path τ without forbidden pairs of arcs, where

(A) 1 precedes n in τ,

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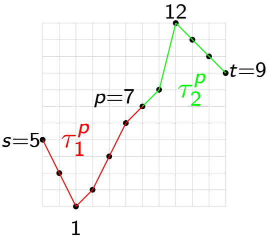

Theorem: The (s,t)-Path-TSP on n cities with a Demidenko distance matrix can be solved in $O((t-s)n^5)$ time for any origin city s and any destination city t, $1 < s < t < n$.

Contents

Van der Veen matrices and pyramidal tours

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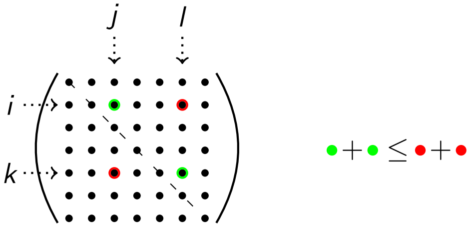

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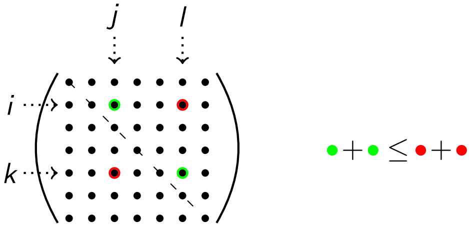

Theorem (Van der Veen, 1994)
For a set of cities with a Van der Veen distance matrix, there exists an optimal tour which is pyramidal. In this case the TSP on n cities can be solved in $O(n^2)$ time by dynamical programming.

Forbidden pairs of arcs

In an (s,t)-path τ a pair of arcs $(i,\tau(i))$ and $(j,\tau(j))$ is called a forbidden pair of arcs of type A or B if the condition (A) or (B) holds

(A) $i < j < \tau(j) < \tau(i)$ or $i > j > \tau(j) > \tau(i)$

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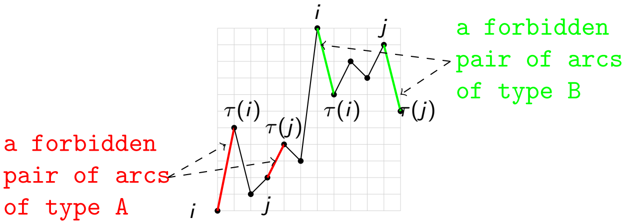

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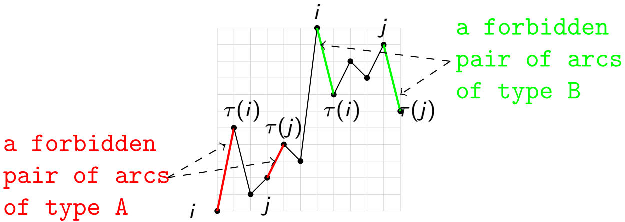

The forbidden pairs lemma: In the case of a Van der Veen distance matrix (c_{ij}), there is an optimal (s,t)-TSP-path without forbidden pairs of arcs for any s and t.

Particular subpaths

Particular subpaths

An arc $(k+1, k)$ in a path $\tau = \langle \ldots, i, k+1, k, j, \ldots \rangle$ with $i = \tau^{-1}(k+1) < \tau(k+1) = k < k+1 < \tau(k) = j$ is called a short zigzag with peak $k+1$ (and valley k).

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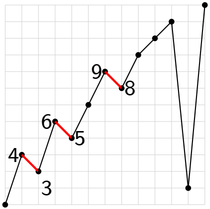

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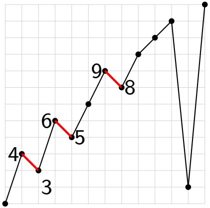

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Short zigzag lemma: Let τ be a $(1,n)$-TSP-path without forbidden pairs of arcs such that $\tau(k+1) = k$ for some $k \in \overline{1, n-1}$. Then $(k+1,k)$ is a short zigzag with peak $k+1$ and $\tau(k) \in \{k+2, k+3\}$, $\tau^{-1}(k+1) \in \{k-2, k-1\}$ hold.

Particular subpaths

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A path $\tau_{p,v} = \langle v+1, v+3, \dots, p-2, p, v, v+2, \dots, p-1\rangle$ with $v < p-1$, and v, p of different parity, $v, p \in \overline{1, n}$, is called a long zigzag with peak p and valley v and is denoted by $\tau_{p,v}$.

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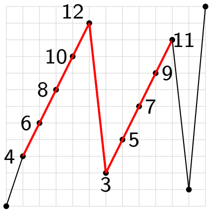

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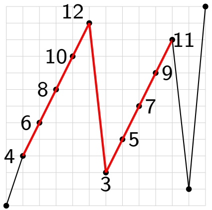

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Long zigzag lemma: Let τ be a $(1, n)$-TSP-path without forbidden pairs of arcs. Let p be a peak and let $v < p-1$ be the first valley reached after p in τ. Then the long zigzag $\tau_{p,v}$ is a subpath of τ. Moreover $\tau_{p,v}$ is preceeded by $x \in \{v-1, v-2\}$ and is succeeded by $y \in \{p+1, p+2\}$.

Particular subpaths

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For two cities i, j with $i, j \in \overline{0, n+1}$, $i < j - 1$ the monotone increasing subpath $\langle i+1, \ldots, j-1 \rangle$ is called a chain between i and j and is denoted by $ch(i, j)$.

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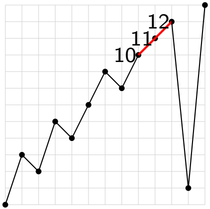

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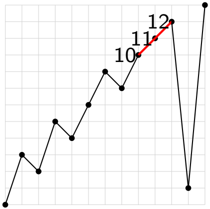

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Zigzag path lemma: Any $(1, n)$-TSP-path without forbidden pairs of arcs is a zigzag path.

Example of a zigzag path

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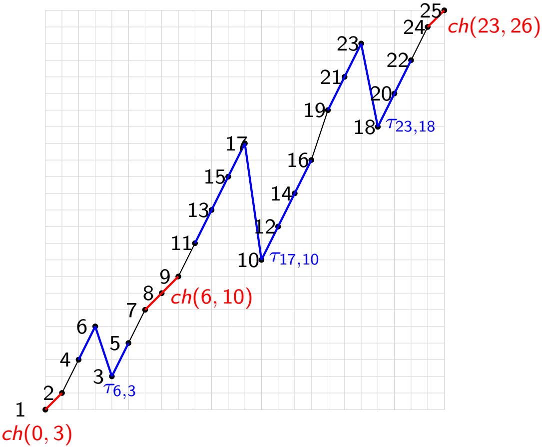

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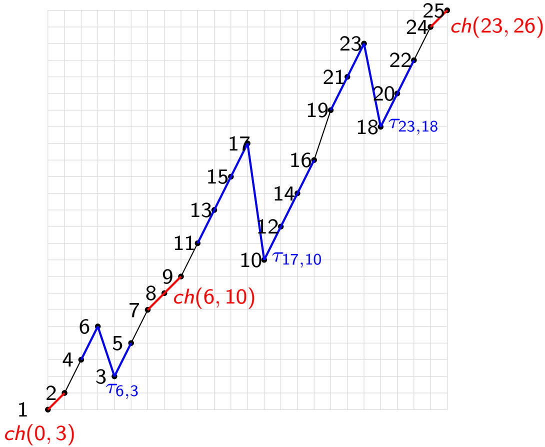

Modelling zigzag paths on a directed graph with $O(n)$ vertices

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The vertex set: $V_n := \{1, 2, \ldots, n\} \cup \{\bar{3}, \bar{4}, \ldots, \overline{n-1}\} \cup \{\underline{2}, \underline{3}, \ldots, \underline{n-2}\}$.

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  - from a zigzag with peak $k+1$ to a zigzag with valley $k+2$: $\left(\underline{k}, \overline{k+3}\right)$
  - from a zigzag with peak $k+1$ to a chain: $\left(\underline{k}, k+2\right)$
  - from a chain to a zigzag with valley $k+1$: $\left(k, \overline{k+2}\right)$

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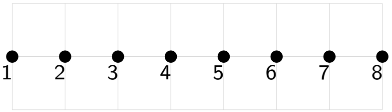

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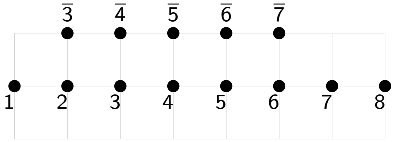

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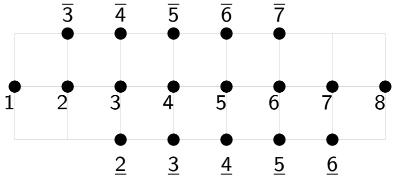

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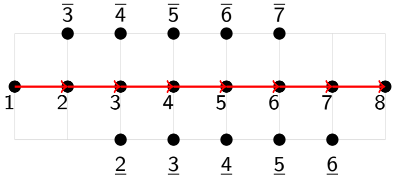

arcs contained in chains

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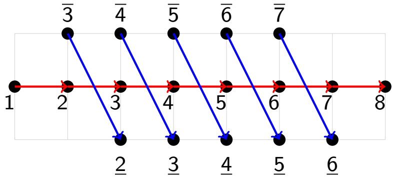

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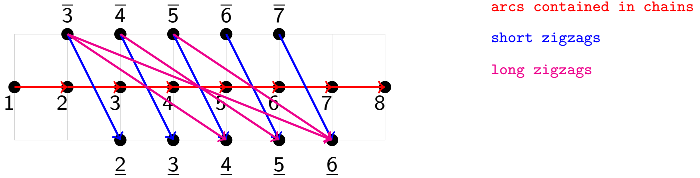

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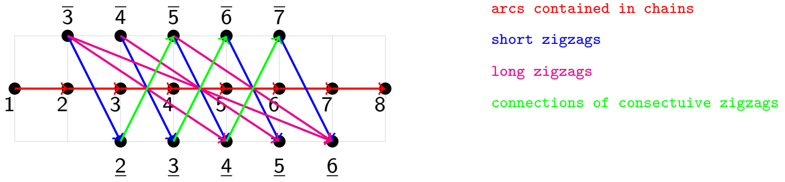

Modelling zigzag paths on a directed graph with $O(n)$ vertices

Observation: The zigzag paths coincide with the $(1, n)$ paths in the auxiliary graph.

Theorem: The shortest $(1, n)$-TSP-path on n cities with a Van der Veen distance matrix can be found in $O(n^2)$ time. Moreover, for each $k \in \overline{1, n}$ a family of shortest $(1, k)$-TSP-path over the cities $\overline{1, k}$ can be found in $O(n^2)$ time.

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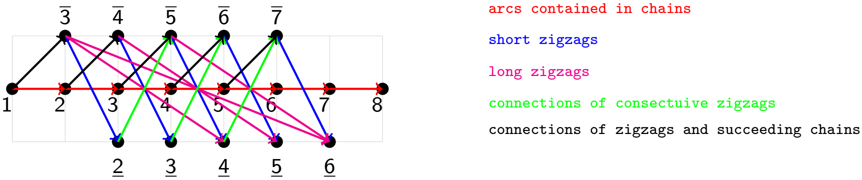

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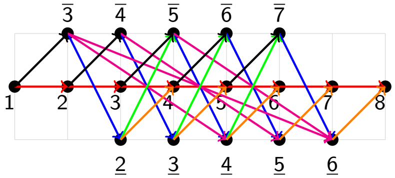

arcs contained in chains

short zigzags

long zigzags

connections of consecutuive zigzags

connections of zigzags and succeeding chains

connections of chains and succeeding zigzags

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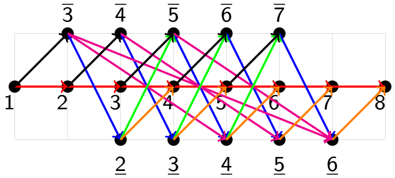

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Theorem: For any $s, t \in \{1, 2, \ldots, n\}$, $1 < s < t < n$, a shortest (s, t)-TSP-path on n cities with a Van der Veen distance matrix can be found in $O(n^3)$ time.

Conclusions

- We discussed two new polynomially solvale special cases of the Path-TSP:
  (I) n cities with a Demidenko distance matrix: solvable in $(t-s)n^5$ for general s and t, solvable in $O(n^5)$ for $s=1$, solvable in $O(n)$ for $s=1$, $t=n$.
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  (II) n cities with a Van der Veen distance matrix: solvable in $O(n^3)$ for general s and t, solvable in $O(n^2)$ if $s=1$.
- The optimal (s,t)-TSP-paths can be found among paths with a particular combinatorial structure avoiding forbidden pairs of arcs.
- The sets of (s,t)-TSP-paths with the above structure are of exponential size.
- The shortest (s,t)-TSP-path over each of the sets above can be determined in polynomial time.

Outlook

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Thanks a lot for the attention!