Forbidden Vertices for some classes of $0-1$ polytopes

Gustavo Angulo[1] **Esteban Salgado**[23]

[1]Department of Industrial and Systems Engineering,
Pontificia Universidad Católica de Chile, Chile

[2]Instituto di Analisi dei Sistemi ed Informatica
"Antonio Ruberti" - CNR, Italy

[3]Department of Computer, Control and Management Engineering
Antonio Ruberti, Sapienza University of Rome, Italy

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Forbidden-vertices

Problem statement

Forbidden-vertices

Given a polytope $X \subseteq \mathbb{R}^n$, a set $V \subseteq \texttt{vert}(X)$, and a vector $c \in \mathbb{R}^n$, the *forbidden-vertices* problem is to either assert $\texttt{vert}(X) \setminus V = \emptyset$, or to return an element in $\texttt{argmin}\{c^\top x \mid x \in \texttt{vert}(X) \setminus V\}$.

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- To approach this problem we want to describe

$$\mathtt{forb}(X, V) := \mathtt{conv}(\mathtt{vert}(X) \setminus V)$$

Complexity of forbidden-vertices

Theorem (G. Angulo, S. Ahmed, S. Dey and V. Kaibel, 2015)
Forbidden-vertices is in general $\mathcal{NP}$-hard except when $X \subseteq \{0,1\}^n$ and both X and V are described explicitly.

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Theorem (G. Angulo, S. Ahmed, S. Dey and V. Kaibel, 2015)
If X is a $0-1$ polytope and for $V \subseteq \mathtt{vert}(X)$

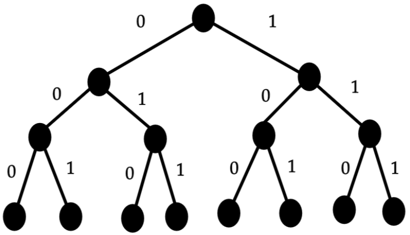
$$\mathtt{xc}(\mathtt{forb}(X,V)) \in \mathcal{O}(n|V|(\mathtt{xc}(X)+1))$$

Extended formulations

Prefixes

Definition

Given a set $V \subseteq \texttt{vert}(X)$ of non-valid vertices, we define the set W of *prefixes* as the vectors of length $i \leq n$ such that any completion $y \in \{0,1\}^{n-i}$ feasible for $\texttt{vert}(X)$ satisfies $(w, y) \in \texttt{vert}(X) \setminus V$.



- $X = [0,1]^3, V = \{0,0,0\}$

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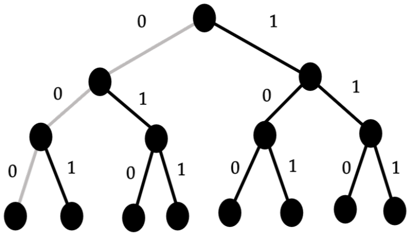

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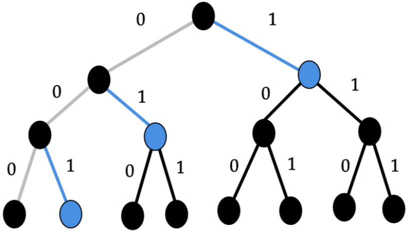

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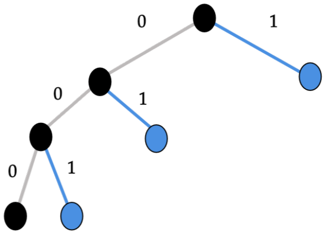

- $X = [0,1]^3, V = \{0,0,0\}$
- $\{(1)\,,\,(0,1)\,,\,(0,0,1)\}$

Prefixes

Definition
Given a set $V \subseteq \texttt{vert}(X)$ of non-valid vertices, we define the set W of *prefixes* as the vectors of length $i \leq n$ such that any completion $y \in \{0,1\}^{n-i}$ feasible for $\texttt{vert}(X)$ satisfies $(w, y) \in \texttt{vert}(X) \setminus V$.

- If $(V^i, X^i)_{1 \leq i \leq n}$ are the projections of V and X onto the first i components

$$W = (X^1 \setminus V^1) \cup \bigcup_{i=2}^{n} ([(V^{i-1} \times \{0,1\}) \cap X^i] \setminus V^i)$$

Classes of 0-1 polytopes

$(\leq k)$-Simplex

$$X_k^n = \{x \in [0,1]^n \mid 1^\top x \leq k\}$$

$s - t$ path polytope

$$X_{st}^G = \{x \in [0,1]^{E(G)} \mid x \text{ is a } s - t \text{ path in } G\}$$

$(\leq k)$-Simplex

- Let W_{ib} be the set of prefixes w of length i with $1^\top w = k - b$.
- Let $\mathcal{T} := \{(i,b) \mid W_{ib} \neq \emptyset\}$.
- $\min\{c^\top x \mid x \in \texttt{forb}(X_k^n, V)\}$ is equivalent to

$$\gamma = \min_{(i,b)\in\mathcal{T}} \left\{ \underbrace{\min_{w\in W_{ib}} \sum_{j=1}^{i} c_j w_j}_{=:\alpha_{ib}} + \underbrace{\min_{x\in X_b^{n-i}} \sum_{j=i+1}^{n} c_j x_{j-i}}_{=:\beta_{ib}} \right\}.$$

- α_{ib} is the objective value of the best prefix with i entries and $(k-b)$ ones.
- β_{ib} is the objective value of the best completion, which is obtained by optimizing over a $(\leq b)$–Simplex in dimension $n-i$.

$(\leq k)$-Simplex

$$\gamma = \min_{(i,b)\in\mathcal{T}} \left\{ \underbrace{\min_{w\in W_{ib}} \sum_{j=1}^{i} c_j w_j}_{=:\alpha_{ib}} + \underbrace{\min_{x\in X_b^{n-i}} \sum_{j=i+1}^{n} c_j x_{j-i}}_{=:\beta_{ib}} \right\}.$$

$$\begin{pmatrix} 1 \\ 1 \\ 0 \\ \star \\ \vdots \\ \star \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 1 \\ \star \\ \vdots \\ \star \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 1 \\ \star \\ \vdots \\ \star \end{pmatrix}$$

$$\alpha_{3(k-2)} = \min\{ c_1 + c_2 , c_1 + c_3 , c_2 + c_3 \}$$

$$\beta_{3(k-2)} = \min\left\{ c^\top x \;\middle|\; x \in X_{k-2}^{n-3} \right\}$$

$(\leq k)$-Simplex - LP formulation

The problem is equivalent to $\max\{\gamma \,|\, (1)\}$

$$\gamma \leq \alpha_{ib} + \beta_{ib} \qquad (\forall\,(i,b) \in \mathcal{T}) \qquad (\varphi_{ib}) \tag{1a}$$

$$\alpha_{ib} \leq \sum_{j=1}^{i} c_j w_j^{ibh} \qquad (\forall\,(i,b) \in \mathcal{T},\, h \leq \ell_{ib}) \qquad (\pi_{ibh}) \tag{1b}$$

$$\beta_{ib} \leq \beta_{(i+1)b} \qquad (\forall\, 1 \leq i \leq n-1,\, 0 \leq b \leq k) \qquad (\sigma_{ib}) \tag{1c}$$

$$\beta_{ib} \leq c_{(i+1)} + \beta_{(i+1)(b-1)} \qquad (\forall\, 1 \leq i \leq n-1,\, 1 \leq b \leq k) \qquad (\eta_{ib}) \tag{1d}$$

$$\beta_{nb} = 0 \qquad (\forall\, 0 \leq b \leq k) \qquad (\delta_{nb}) \tag{1e}$$

$$\beta_{i0} = 0 \qquad (\forall\, 1 \leq i \leq n) \qquad (\varepsilon_{i0}) \tag{1f}$$

$(\leq k)$-Simplex - Dual LP formulation

The dual is to minimize $\sum_{j=1}^{n} c_j \left[\sum_{\substack{(i,b)\in\mathcal{T} \\ 1\leq h\leq |W_{ib}|, 1\leq j\leq i}} w_j^{ibh}\pi_{ibh} + \chi_{(2\leq j)} \sum_{1\leq b\leq k} \eta_{(j-1)b} \right]$

subject to

$$\varphi_{ib} - \sum_{1\leq h\leq |W_{ib}|} \pi_{ibh} = 0 \qquad (\forall (i,b)\in\mathcal{T}) \tag{2a}$$

$$\sum_{(i,b)\in\mathcal{T}} \varphi_{ib} = 1 \tag{2b}$$

$$F_{ib}(\varphi,\pi,\sigma,\eta,\delta,\varepsilon) = 0 \qquad (\forall 1\leq i\leq n,\ 0\leq b\leq k) \tag{2c}$$

$$\varphi,\pi,\sigma,\eta \geq 0 \tag{2d}$$

$(\leq k)$-Simplex

- By strong duality (2) is equivalent to $\min\{c^\top x \mid x \in \texttt{forb}(X_k^n, V)\}$.

Theorem (G. Angulo, E.S.)
Constraints (2a)-(2d) coupled with

$$x_j = \sum_{\substack{(i,b)\in\mathcal{T} \\ 1\leq h\leq |W_{ib}|, 1\leq j\leq i}} w_j^{ibh}\pi_{ibh} + \chi_{(2\leq j)} \sum_{1\leq b\leq k} \eta_{(j-1)b} \; (\forall 1 \leq j \leq n)$$

define an extended formulation for $\texttt{forb}(X_k^n, V)$ of size $\mathcal{O}(n^2 + n|V|)$

- This bound is better than the previously known $\mathcal{O}(n^2|V| + n|V|)$.

$s - t$ path polytope

- Let $G = (N, A)$ be a DAG where every node and arc is in some $s - t$ path, with nodes sorted in a topological order $\{s = v_1, \dots, v_n = t\}$ and arcs sorted in lexicographical order.
- Let W_{iv} be the set of prefixes w of length i that describe an $s - v$ path.
- Let $\mathcal{T} := \{(i, v) \mid W_{iv} \neq \emptyset\}$.
- $\min\{c^\top x \mid x \in \texttt{forb}(X_{st}^G, V)\}$ is equivalent to

$$\gamma = \min_{(i,v)\in\mathcal{T}} \left\{ \underbrace{\min_{w\in W_{iv}} \sum_{j=1}^{i} c_j w_j}_{=:\alpha_{iv}} + \underbrace{\min_{x\in X_{v,t}} \sum_{j=i+1}^{m} c_j x_{j-i}}_{=:\beta_{iv}} \right\}.$$

$s - t$ path polytope - LP formulation

The problem is equivalent to $\max\{\gamma \,|\, (3)\}$

$$\gamma \le \alpha_{iv} + \beta_{iv} \qquad (\forall\,(i,v) \in \mathcal{T}) \qquad (\varphi_{iv}) \qquad (3a)$$

$$\alpha_{iv} \le \sum_{j=1}^{i} c_j w_j^{ivh} \qquad (\forall\,(i,v) \in \mathcal{T},\ h \le \ell_{iv}) \qquad (\pi_{ivh}) \qquad (3b)$$

$$\beta_{iv} \le \beta_{(i+1)v} \qquad (\forall\, 1 \le i \le m-1,\ s \le v < t) \qquad (\sigma_{iv}) \qquad (3c)$$

$$\beta_{iv} \le c_{(i+1)} + \beta_{(i+1)w} \qquad (\forall\, 1 \le i \le m-1,\ s \le v < t,\ w \in \delta^+(v)) \qquad (\eta_{ivw}) \qquad (3d)$$

$$\beta_{iv} = 0 \qquad (\forall\, i = m,\ s \le v \le t) \qquad (\delta_{iv}) \qquad (3e)$$

$$\beta_{iv} = 0 \qquad (\forall\, 1 \le i \le m,\ v = t) \qquad (\varepsilon_{iv}) \qquad (3f)$$

$s - t$ path polytope - Dual LP formulation

The dual is to minimize $\sum_{j=1}^{n} c_j \left[\sum_{\substack{(i,v)\in\mathcal{T} \\ h,j\leq i}} w_j^{ivh} \pi_{ivh} + \chi_{(2\leq j)} \sum_{\substack{v<t \\ w\in\delta^+(v)}} \eta_{(j-1)vw} \right]$

subject to

$$\varphi_{iv} - \sum_{1\leq h\leq |W_{iv}|} \pi_{ivh} = 0 \qquad (\forall (i,v)\in\mathcal{T}) \tag{4a}$$

$$\sum_{(i,v)\in\mathcal{T}} \varphi_{iv} = 1 \tag{4b}$$

$$F_{iv}(\varphi,\pi,\sigma,\eta,\delta,\varepsilon) = 0 \qquad (\forall\, 1\leq i\leq m,\ s\leq v\leq t) \tag{4c}$$

$$\varphi,\pi,\sigma,\eta \geq 0 \tag{4d}$$

$s - t$ path polytope

- By strong duality (4) is equivalent to $\min\{c^\top x \mid x \in \texttt{forb}(X_{st}^G, V)\}$.

Theorem (G. Angulo, E.S.)
Constraints (4a)-(4d) coupled with

$$x_j = \sum_{\substack{(i,v)\in\mathcal{T} \\ h,j\leq i}} w_j^{ivh}\pi_{ivh} + \chi_{(2\leq j)} \sum_{\substack{v<t \\ w\in\delta^+(v)}} \eta_{(j-1)vw}$$

define an extended formulation for $\texttt{forb}(X_{st}^G, V)$ of size $\mathcal{O}(|A||N| + |A|^2)$

- This bound is better than the previously known $\mathcal{O}(|A||N| + |A|^2|N|)$.

Numerical results

Describing the experiment

- Instances for *Prize collecting TSP* (PCTSP) in which the objective is to find an optimal cycle of length at least k in a complete directed graph with n nodes, with a depot node in which the cycle must start, with costs for using an edge and costs for not using a node

$$\min\{c^\top x + d^\top y \,|\, (x, y) \in P \cap (\{0, 1\}^{n(n-1)} \times \texttt{vert}(X^n_{n-k}))\}$$

Describing the experiment

- We compare three formulations:

  1. $\min\{c^\top x + d^\top y \,|\, (x,y) \in P \cap [\{0,1\}^{n(n-1)} \times \texttt{forb}(X^n_{n-k}, V)]\}$ with the extended formulation previously described.

  2. $\min\{c^\top x + d^\top y \,|\, (x,y) \in P \cap [\{0,1\}^{n(n-1)} \times (\texttt{vert}(X^n_{n-k}) \cap N(V))]\}$ where $N(V)$ is described by the constraints $\sum_{i:v_i=1}(1-y_i) + \sum_{i:v_i=0} y_i \geq 1$ for every $v \in V$. These are the *no-good-cut* constraints.

  3. $\min\{c^\top x + d^\top y \,|\, (x,y) \in P \cap [\{0,1\}^{n(n-1)} \times (\texttt{forb}([0,1]^n, V) \cap X^n_{n-k})]\}$ with the extended formulation described in *G. Angulo et al.*

Results

Table 1: Comparative results for the three formulations.

Instance		No-good-cut			$\texttt{forb}([0,1]^n, V) \cap X^n_{n-k}$			$\texttt{forb}(X^n_{n-k}, V)$		
N	k	gap(%)	time opt(s)	time rel(s)	gap(%)	time opt(s)	time rel(s)	gap(%)	time opt(s)	time rel(s)
	0	1.43	**69.964**	**19.772**	**1.31**	73.604	21.109	**1.31**	72.067	21.067
	10	1.43	**81.768**	**18.721**	**1.31**	83.906	20.648	**1.31**	85.506	19.994
40	20	1.42	**57.983**	**19.013**	1.31	67.878	20.621	**1.30**	78.654	20.639
	30	1.43	98.193	21.071	**1.31**	78.998	**19.053**	**1.31**	**64.777**	20.961
	39	1.42	115.945	24.589	**1.31**	105.057	24.092	**1.31**	**88.616**	**22.154**
	0	2.08	**217.627**	101.232	**2.02**	239.063	**91.142**	**2.02**	244.419	91.328
	12	2.08	**228.860**	90.088	**2.02**	236.479	88.152	**2.02**	229.157	**84.561**
50	25	2.08	219.868	96.456	**2.02**	218.359	91.988	**2.02**	**216.824**	**89.751**
	37	2.08	274.350	94.180	**2.02**	246.844	**91.091**	**2.02**	**219.868**	96.456
	49	2.08	250.276	93.379	**2.02**	264.533	91.729	**2.02**	**249.530**	**90.112**

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