Comparing Formulations for Piecewise Convex Problems

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Introduction

$$\min \sum_{j \in N} c_j x_j \quad (1)$$

$$f_i(x) + \sum_{j \in H(i)} g_{ij}(x_j) \leq 0 \qquad i \in M \quad (2)$$

$$l_j \leq x_j \leq u_j \qquad j \in N \quad (3)$$

$$x_j \in \mathbb{Z} \qquad j \in I. \quad (4)$$

- Non-convex Mixed-Integer Non-Linear Programs where the non-convexity is manifested as the sum of non-convex univariate functions:
  - The sets M, N, $I \subseteq N$, and $H(i) \subseteq N$ are finite;
  - The functions $f_i : \mathbb{R}^n \to \mathbb{R}$ are convex and multivariate;
  - The functions $g_{ij} : \mathbb{R} \to \mathbb{R}$ are non-convex univariate.
  - We assume that l_j and u_j are finite bounds for x_j that appear in g_{ij} functions.

Introduction

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$$f_i(x) + \sum_{j \in H(i)} g_{ij}(x_j) \leq 0 \qquad i \in M \tag{2}$$

$$l_j \leq x_j \leq u_j \qquad j \in N \tag{3}$$

$$x_j \in \mathbb{Z} \qquad j \in I. \tag{4}$$

- For this simplified notation, we can exploit more forms:
  - $H(i) = \emptyset$, means that the ith constrait is convex, and $f(i)$ may well be linear;
  - The objective function can also have the $f_i(x) + \sum_{j \in H(i)} g_{ij}(x_j)$ form;
  - Not all variables need necessarily appear in some nonconvex term:
    - The bounds $l_j \leq x_j \leq u_j$ are not necessary for this case;

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Sequential Convex MINLP

- Sequential Convex MINLP (SC-MINLP) technique has been defined in [D'Ambrosio et al., 2012];
- Computing a piecewise-convex relaxation of each $g_{ij}(x_j)$;
- The breakpoints $l_j = l_{ij}^1 < l_{ij}^2 < ... < l_{ij}^{s(ij)} < l_{ij}^{s(ij)+1} = u_j$, where the non-convex functions g_{ij} change convexity/concavity;
  - In practice, this is done by computing by the zeros of the second derivative of g_{ij} using some algebraic package, such as MATLAB.
- Then, for fixed i and $j \in H(i)$, we denote by:
  - $S_{ij} = \{\, s \,:\, g_{ij}$ is *concave* in the sub-interval $[l_{ij}^s, l_{ij}^{s+1}]\,\}$, and
  - $\check{S}_{ij} = \{\, s \,:\, g_{ij}$ is *convex* in the sub-interval $[l_{ij}^s, l_{ij}^{s+1}]\,\}$.
- On S_{ij} the function is substituted with its the best possible convex relaxation (a linear function), while the convex parts are kept as they are.

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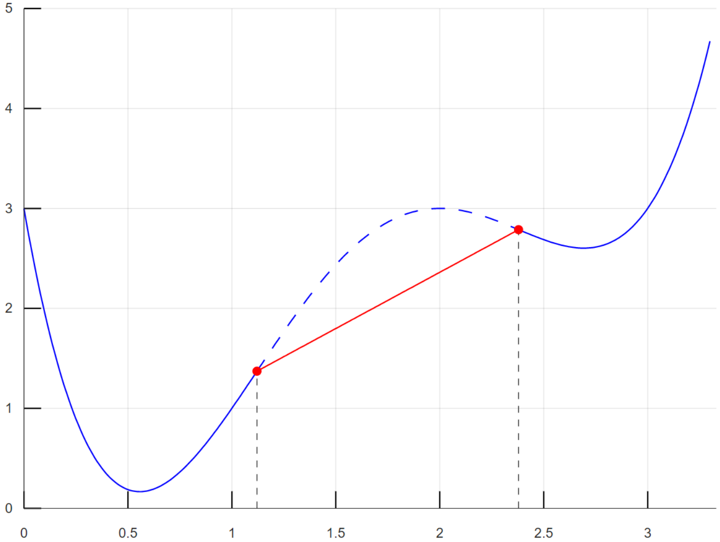
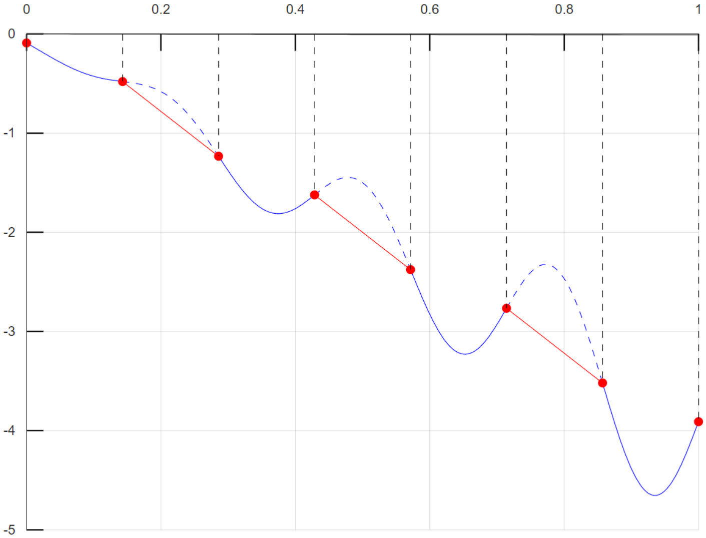
Example 1

$$(x-2)^4 + (x-3)^3 + (x-4)^2 + x - 2$$



Example 2

$$-0.0909457cos(21.9912x) - sin(21.9912x)x - 4x$$



Sequential Convex MINLP

- This defines a *convex* MINLP, whose continuous relaxation therefore provides valid lower bounds.
- This reformulation step can actually be done in different ways:
  - *Incremental Model*;
  - *Convex-Combination*;
  - *Multiple Choice*;
- For Mixed-Integer Linear Programs, since they have similar size and provide the same lower bound [Croxton et al., 2003];
  - However, this is not the case for nonlinear piecewise-convex programs.

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Incremental Model (IM)

$$\min \sum_{j\in N} c_j x_j \tag{5}$$

$$\bar{f}_i(x) + \sum_{j\in H(i)} \sum_{s\in \check{S}(ij)} z_{ij}^s \le 0 \qquad i \in M \tag{6}$$

$$z_{ij}^s \ge g_{ij}(l_{ij}^s + x_{ij}^s) - g_{ij}(l_{ij}^s) \qquad s \in \check{S}(ij),\ j \in H(i),\ i \in M \tag{7}$$

$$x_j = l_j + \sum_{s\in S(ij)} x_{ij}^s \qquad j \in H(i), i \in M \tag{8}$$

$$(l_{ij}^{s+1} - l_{ij}^s) y_{ij}^{s+1} \le x_{ij}^s \le (l_{ij}^{s+1} - l_{ij}^s) y_{ij}^s \qquad s \in S(ij),\ j \in H(i),\ i \in M \tag{9}$$

$$y_{ij}^s \in \{0,1\} \qquad s \in S(ij),\ j \in H(i),\ i \in M \tag{10}$$

$$x_j \in \mathbb{Z} \qquad j \in I \tag{11}$$

- where $\bar{f}_i = f_i(x) + \sum_{j\in H(i)} g_{ij}(l_{ij}^1) + \sum_{s\in \hat{S}(ij)} \alpha_{ij}^s x_{ij}^s$
- The IM introduces a segment load variable, x_{ij}^s, for each sub-interval $[l_{ij}^s, l_{ij}^{s+1}]$;
- Feasibility requires that the value on x_{ij}^{s+1} be zero unless x_{ij}^s is "full";
- Binary variables, y_{ij}^s, defined by the condition that $y_{ij}^s = 1$ if $x_{ij}^s > 0$, and $y_{ij}^s = 0$ otherwise.

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Multiple Choice Model (MCM)

$$\min \sum_{j \in N} c_j x_j \tag{12}$$

$$\bar{f}_i(x) + \sum_{j \in H(i)} \sum_{s \in \check{S}(ij)} z_{ij}^s \leq 0 \qquad i \in M \tag{13}$$

$$z_{ij}^s \geq g_{ij}(x_{ij}^s) - g_{ij}(0) \qquad s \in \check{S}(ij),\ j \in H(i),\ i \in M \tag{14}$$

$$x_j = \sum_{s \in S(ij)} x_{ij}^s \qquad j \in H(i), i \in M \tag{15}$$

$$l_{ij}^s y_{ij}^s \leq x_{ij}^s \leq l_{ij}^{s+1} y_{ij}^s \qquad s \in S(ij),\ j \in H(i),\ i \in M \tag{16}$$

$$\sum_{s \in S(ij)} y_{ij}^s = 1 \qquad i \in M,\ j \in H(i) \tag{17}$$

$$y_{ij}^s \in \{0, 1\} \qquad s \in S(ij),\ j \in H(i),\ i \in M \tag{18}$$

$$x_j \in \mathbb{Z} \qquad j \in I \tag{19}$$

- $\bar{f}_i = f_i(x) + \sum_{j \in H(i)} g_{ij}(0) \sum_{s \in \check{S}(ij)} y_{ij}^s + \sum_{s \in \hat{S}(ij)} (\alpha_{ij}^s x_{ij}^s + (g_{ij}(l_{ij}^s) - \alpha_{ij}^s l_{ij}^s) y_{ij}^s)$
- The load variable x_{ij}^s, for each segment s, defines the total load $x_{ij}^s = x_j$ and $y_{ij}^{s+1} = 1$, if x_j lies on the sub-interval $[l_{ij}^s, l_{ij}^{s+1}]$. Otherwise, $x_{ij}^s = y_{ij}^{s+1} = 0$.
- In this formulation, at most one y_{ij}^{s+1} will equal one.

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Convex Combination Model (CCM)

$$
\begin{array}{llr}
\min \ \sum_{j\in N} c_j x_j & & (20) \\
\bar{f}_i(x) + \sum_{j\in H(i)} \sum_{s\in \check{S}(ij)} z_{ij}^s \le 0 & i \in M & (21) \\
z_{ij}^s \ge g_{ij}(l_{ij}^s \mu_{ij}^s + l_{ij}^{s+1}\lambda_{ij}^s) - g_{ij}(0) & s \in \check{S}(ij),\ j \in H(i),\ i \in M & (22) \\
x_j = \sum_{s\in S(ij)} (l_{ij}^s \mu_{ij}^s + l_{ij}^{s+1}\lambda_{ij}^s) & j \in H(i), i \in M & (23) \\
\mu_{ij}^s + \lambda_{ij}^s = y_{ij}^s & s \in S(ij),\ j \in H(i),\ i \in M & (24) \\
\sum_{s\in S(ij)} y_{ij}^s = 1 & i \in M,\ j \in H(i) & (25) \\
y_{ij}^s \in \{0,1\} & s \in S(ij),\ j \in H(i),\ i \in M & (26) \\
x_j \in \mathbb{Z} & j \in I & (27)
\end{array}
$$

- $\bar{f}_i = f_i(x) + \sum_{j\in H(i)} g_{ij}(0) \sum_{s\in \check{S}(ij)} (\mu_{ij}^s + \lambda_{ij}^s) + \sum_{s\in \hat{S}(ij)} (g_{ij}(l_{ij}^s)\mu_{ij}^s + g_{ij}(l_{ij}^{s+1})\lambda_{ij}^s)$.
- Considers the variable that x_{ij}^s defines the total load, as in the MCM;
- The load and its cost are computed as a convex combination of the load/cost of the two endpoints of the segment;
  - Defining multipliers μ_{ij}^s and λ_{ij}^s as the weights of these two endpoints.

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Computational Experiments

- We focus in two different problems:
  - Non linear knapsack problem;
  - Aircraft Conflict Avoidance (working in progress);
- For lack of space, we focus on the latter and compare the three formulations;

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Non linear knapsack problem

- The non linear knapsack problem is the same considered in [D'Ambrosio et al., 2019], i.e.:

$$\max \sum_{j \in N} p_j$$

$$p_j \leq \frac{c_j}{1 + b_j \exp(-a_j(x_j + d_j))} \qquad j \in N$$

$$\sum_{j \in N} x_j \leq C$$

$$0 \leq x_j \leq U_j \qquad j \in N$$

- For each value of $|N| \in \{10, 20, 50, 100\}$ we randomly generated 10 instances, where $a_j \in [0.1, 0.2]$, $b_j \in [0, 100]$, $c_j \in [0, 100]$, and $d_j \in [-100, 0]$ were uniformly drawn in the corresponding intervals. We fixed $U_j = 100$ for all $j \in N$ and $C = 100|N|/2$.

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Non linear knapsack problem

Table: Computational results for non linear knapsack problem.

Instance	Original	Incremental		Multiple Choice		Convex Combination	
Items	Solution	Integer	Relax	Integer	Relax	Integer	Relax
10	193.372	196.034	205.313	196.035	201.773	196.035	201.773
20	250.601	251.077	274.085	251.079	259.449	251.079	259.449
50	337.435	337.997	394.567	337.652	344.921	338.001	345.098
100	422.279	422.597	504.163	422.162	429.720	414.255	418.741
Average	299.985	301.001	343.249	300.811	308.044	298.864	305.269

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Aircraft Conflict Avoidance

- This problem is the same considered in [Cafieri and Omheni, 2017]:
- The relative initial distance $X_{ij}^{r_0}$ between aircraft i and j and the relative speed V_{ij}^{r} are vectors in $\mathbb{R}^2$.

$$X_{ij}^{r_0} := \begin{pmatrix} x_i^0 - x_j^0 \\ y_i^0 - y_j^0 \end{pmatrix} \tag{28}$$

$$V_{ij}^{r} := \begin{pmatrix} \cos(\phi_i + \theta_i)v_i - \cos(\phi_j + \theta_j)v_j \\ \sin(\phi_i + \theta_i)v_i - \sin(\phi_j + \theta_j)v_j \end{pmatrix} \tag{29}$$

Aircraft Conflict Avoidance

- For $B := \{(i,j) \in A \times A : i < j\}$, the optimization problem is defined below:

$$\min \sum_{i \in A} \theta_i^2 \tag{30}$$

s.t.:

$$\theta_i^{min} \leq \theta_i \leq \theta_i^{max} \qquad \forall i \in A, \tag{31}$$

$$y_{ij}(||V_{ij}^r||^2(||X_{ij}^{r_0}||^2 - d^2) - (X_{ij}^{r_0}.V_{ij}^r)^2) \geq 0 \qquad \forall (i,j) \in B, \tag{32}$$

$$t_{ij}^m = -\frac{X_{ij}^{r_0}.V_{ij}^r}{||V_{ij}^r||^2} \qquad \forall (i,j) \in B, \tag{33}$$

$$t_{ij}^m(2y_{ij} - 1) \geq 0, \qquad \forall (i,j) \in B, \tag{34}$$

$$y_{ij} \in \{0,1\}, \qquad \forall (i,j) \in B \tag{35}$$

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Aircraft Conflict Avoidance

- For a reformulated formulation, the optimization problem is defined below:

$$(\mathrm{M}_3)\ \min\ F \tag{36}$$

s.t.:

$$F \geq \sum_{i \in A} \theta_i^2 \tag{37}$$

$$F(\theta_i) + G(\theta_j) + H'_{\Phi_1} + H'_{\Phi_2} + H'_C \geq -M(1 - y_{ij}) \quad \forall (i,j) \in B, \tag{38}$$

$$-M(1 - y_{ij}) \leq -X_{ij}^{r_0}.V_{ij}^{r} \leq M y_{ij}, \quad \forall (i,j) \in B, \tag{39}$$

$$\theta_i^{min} \leq \theta_i \leq \theta_i^{max} \quad \forall i \in A, \tag{40}$$

$$y_{ij} \in \{0, 1\}, \quad \forall (i,j) \in B \tag{41}$$

Where,

$$F(\theta_i) = -(\cos(\phi_i + \theta_i)C' v_i)^2 - (\sin(\phi_i + \theta_i)C'' v_i)^2 - \cos(\phi_i + \theta_i)\sin(\phi_i + \theta_i)2C'C'' v_i^2$$

$$G(\theta_j) = -(\cos(\phi_j + \theta_j)C' v_j)^2 - (\sin(\phi_j + \theta_j)C'' v_j)^2 - \cos(\phi_j + \theta_j)\sin(\phi_j + \theta_j)2C'C'' v_j^2$$

$$H'_{\Phi_1} = \cos(\Phi_1) v_i v_j (C'^2 + C''^2 - 2C)$$

$$H'_{\Phi_2} = \cos(\Phi_2) v_i v_j (C'^2 - C''^2) + \sin(\Phi_2) 2 v_i v_j C' C''$$

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Aircraft Conflict Avoidance

- Working in progress;
- There is more than one way to rewrite the model in univariate functions;
- In practice the Incremental and Multiple Choice models have a more efficient behavior for this problem.

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Conclusions

- In general, the continuous relaxations of MCM and CCM are equivalent
- IM is worse (unless the time limit is reach).
  - However, this does not always imply better CPU times.
- The SC-MINLP technique can be strengthened with the Perspective Reformulation technique;
- We plan to study this aspect more extensively both from a computational and a theoretical viewpoint.

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Thank you!

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