A heuristic for max-cut in toroidal grid graphs

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CTW 2020

Motivation

Statistical Physics

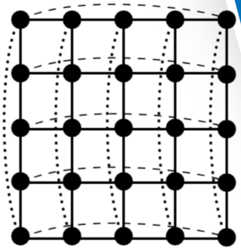
Ground state of spin glasses under the Ising model

Under this model the Hamiltonian of the system is defined by

$$H = -\sum_{\langle i,j \rangle} J_{ij}\sigma_i\sigma_j$$

- σ_i is the i-th spin.
- J_{ij} is the interaction energy between the i-th and the j-th particles.

The goal is to find the lowest energy state.



Max-cut

Max-cut

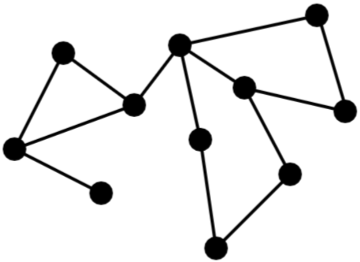
Given $G = (V, E, w)$ the *max-cut problem* calls for a partition $(W : V \setminus W)$ of the node-set defining a maximal-weight edge-cut.

$$\max_{x \in \{-1,1\}^V} \frac{1}{2} \sum_{ij \in E} w_{ij}(1 - x_i x_j)$$

- $x_i = 1 \cdot \chi_{i \in W} - 1 \cdot \chi_{i \in V \setminus W}$

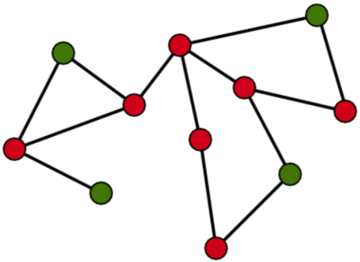
Algorithms: Subgraph sampling scheme

Subgraph sampling scheme



1. Select randomly a point $\hat{x} \in \{-1, 1\}^V$.

Subgraph sampling scheme



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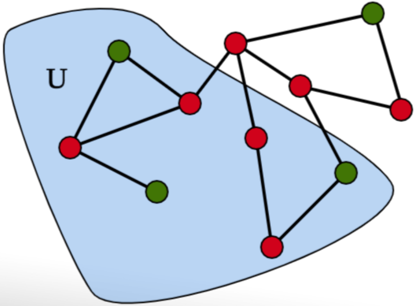

2. Select a “suitable” set $U \subset V$ contracting the nodes $V \setminus U$.

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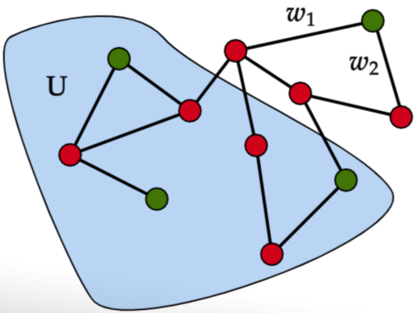

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   - Apply *switching* to the nodes in $V \setminus U$ if needed.

$$\hat{x}_i \leftarrow 1 - \hat{x}_i$$

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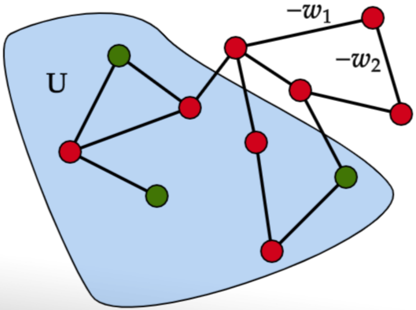

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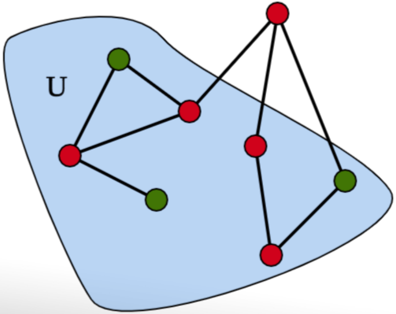

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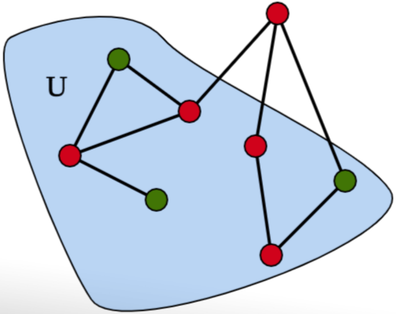
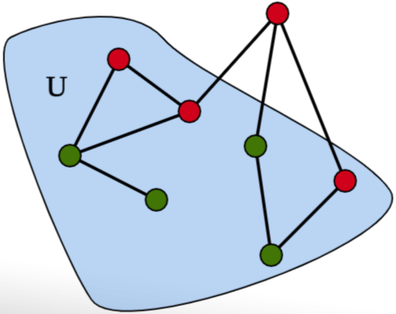


3. Solve max-cut over the "contracted" graph $G_{V \setminus U}$.

Subgraph sampling scheme



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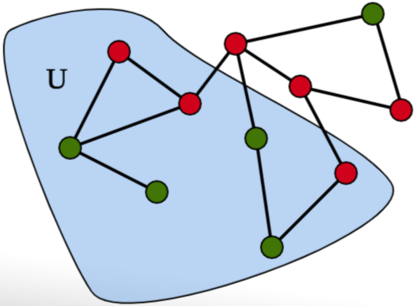

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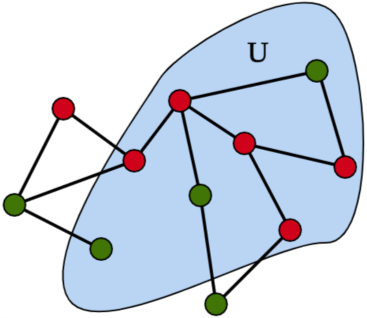
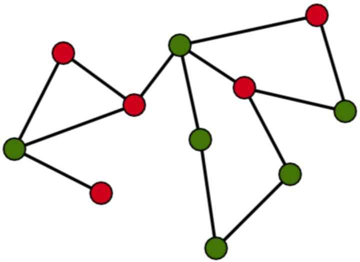


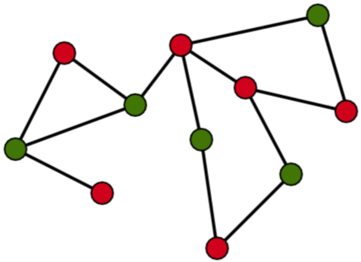
4. Until a maximum number of non-improving iterations is not reached, select a new node-set $U \subset V$ and go to step 2.

Subgraph sampling scheme



5. Perturb the current vector $\hat{x}$ and repeat from step 2. until no more improvements take place.

Subgraph sampling scheme



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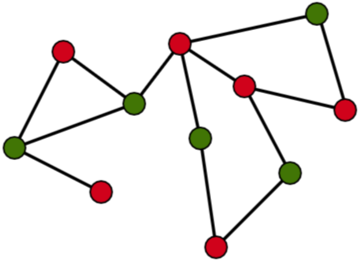
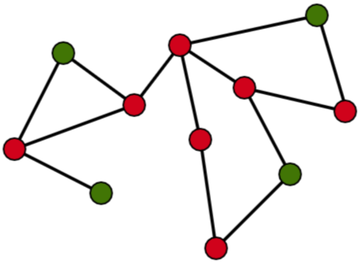


6. If additional computing time is allowed, the node assignment is randomly generated afresh and the process is restarted from step 2.

Subgraph sampling scheme



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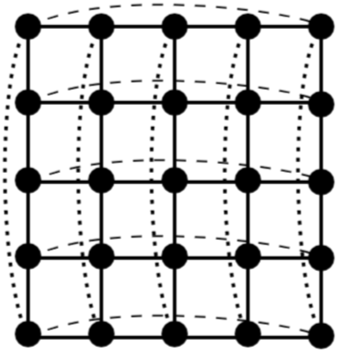
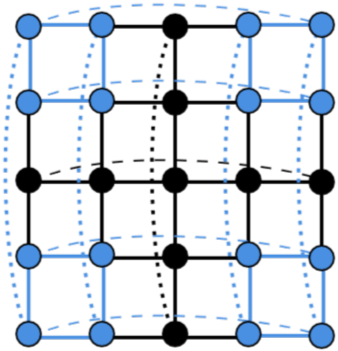
Algorithms: How to select U?

Algorithms: Planar subgraph

Planar subgraph



Planar subgraph



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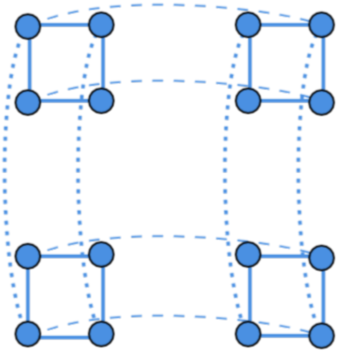

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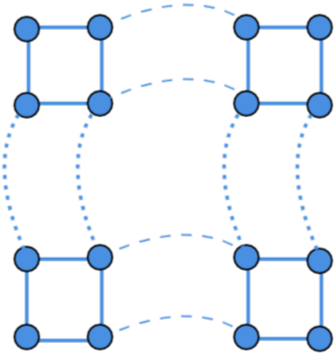

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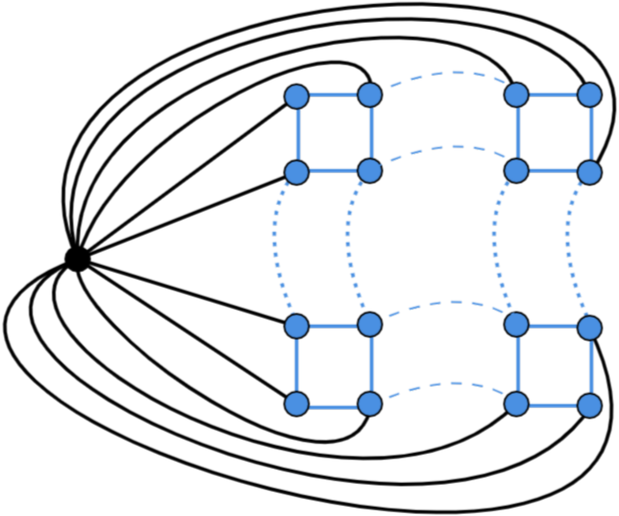
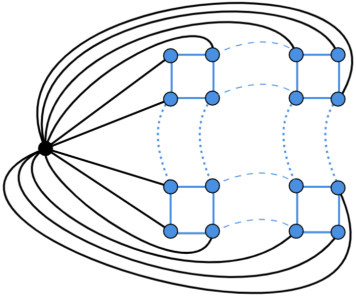


Planar subgraph

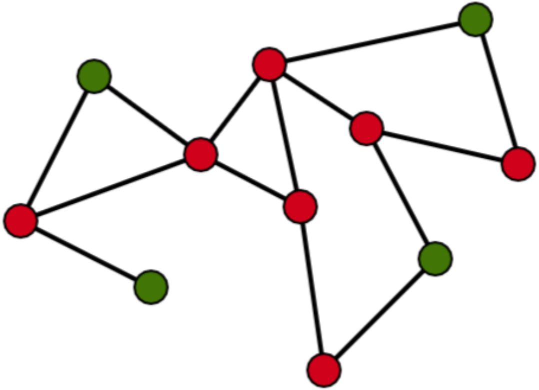
Algorithm (F. Liers and G. Pardella, 2012)

For arbitrary weighted planar graphs max-cut is solvable in $O(|V|^{\frac{3}{2}} \log(|V|))$



Algorithms: Negative subgraph

Negative subgraph - general case



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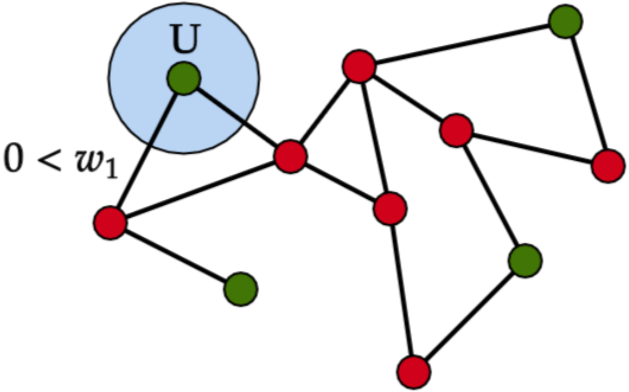

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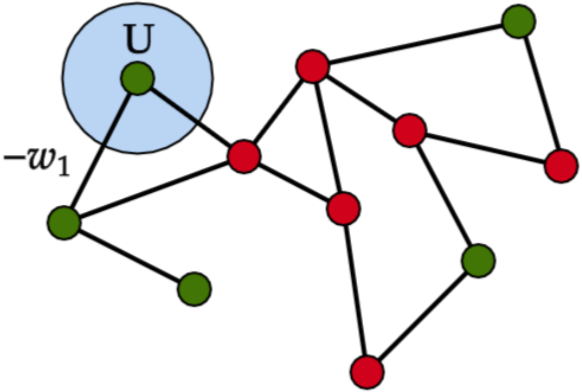

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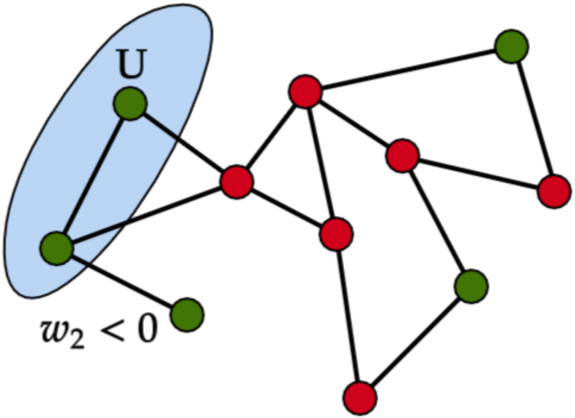

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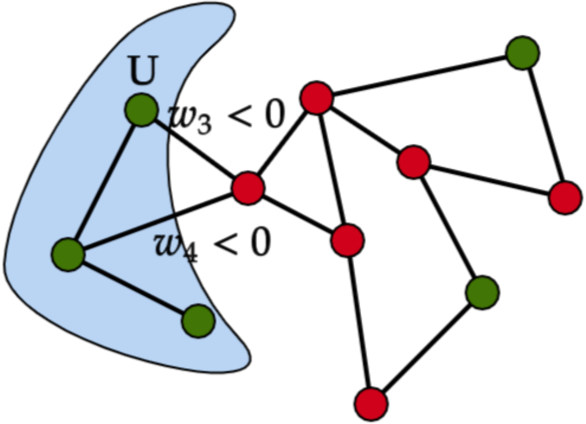

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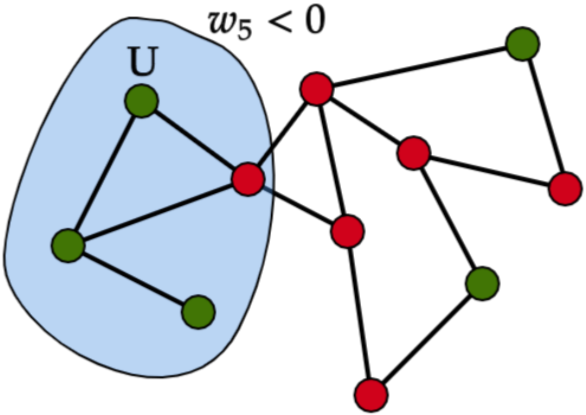

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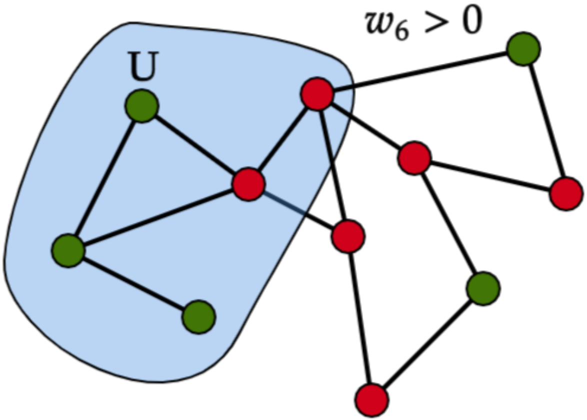

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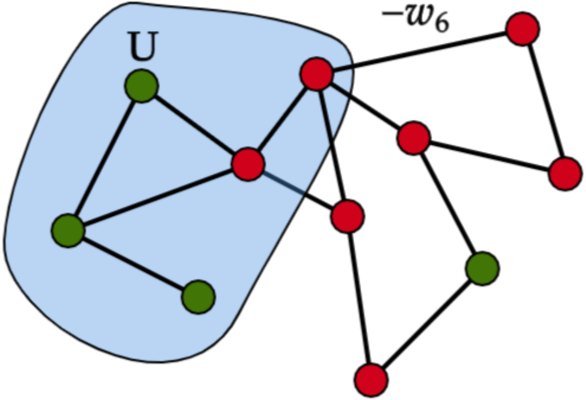

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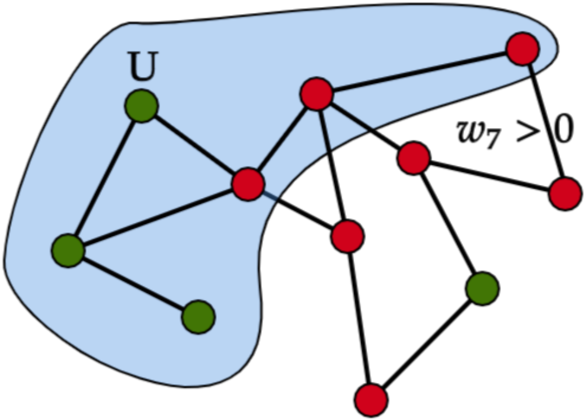

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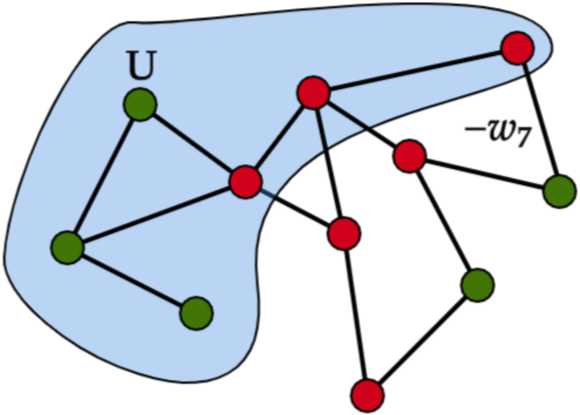

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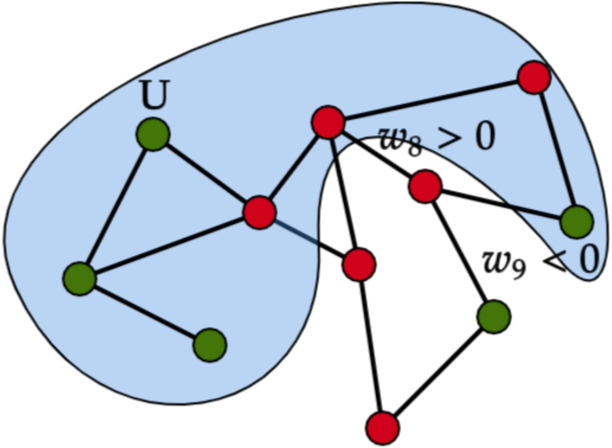

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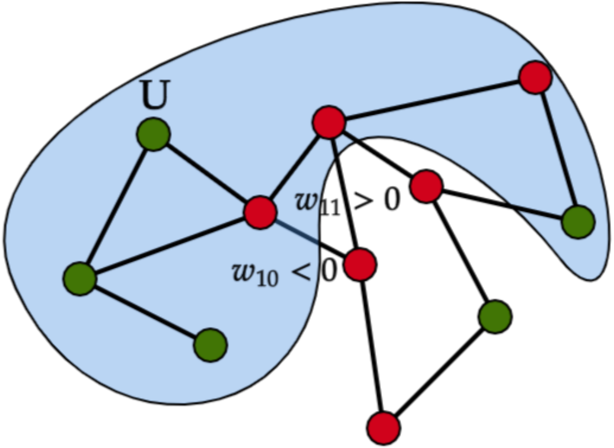

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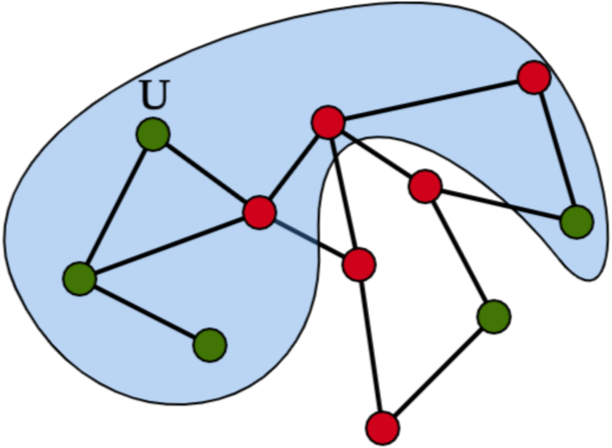
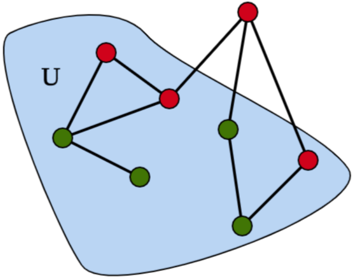
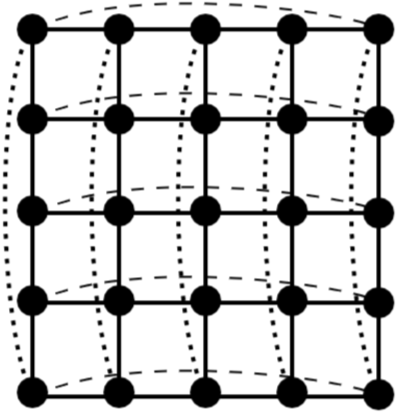


Negative subgraph - general case

Theorem (S.T. McCormick, M.R. Rao and G. Rinaldi, 2003)
For graphs with all the non-negative-weighted edges adjacent to a single node, max-cut is solvable in polynomial time.



Negative subgraph - 2d toroidal grids



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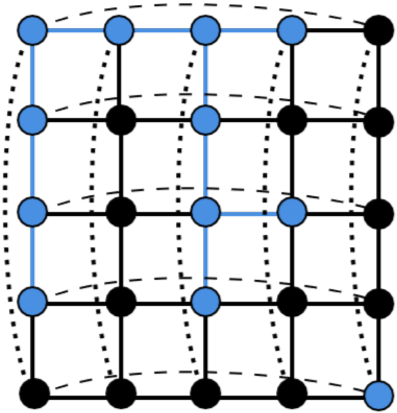

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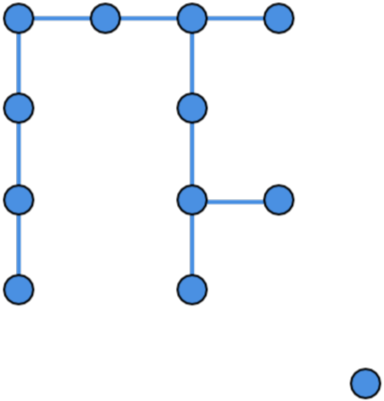

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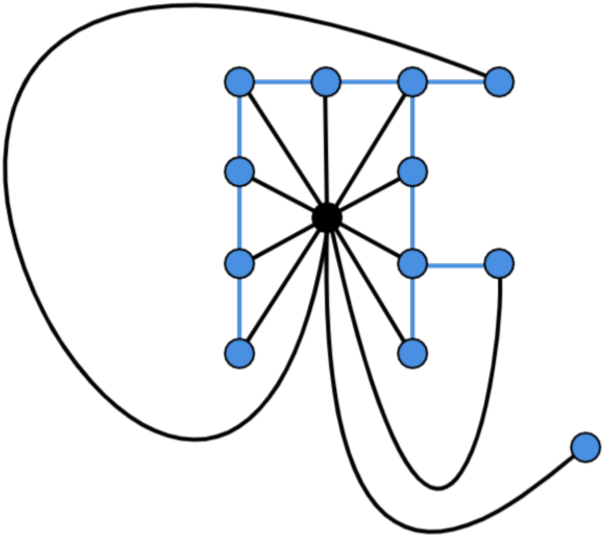

Numerical results

Data for the experiments

- 2d toroidal grids with 316×316 nodes.
- Graphs generated with `rudy` with edge-weights:
  - bivariate ± 1, with proportion p of negative edges.
  - following a standard Gaussian distribution.

Results

		P		No complement				Complement			
p	s	val	time	it (avg)	val (avg)	%difP (avg/std)	time (avg)	it (avg)	val (avg)	%difP (avg/std)	time (avg)
40	1	89968	1612.79	6.70	89770.40	0.22 / 0.02	3681.97	3.20	**89772.60**	0.21 / 0.02	4013.31
	2	89978	1594.09	6.40	89768.80	0.23 / 0.01	3651.78	3.90	**89779.00**	0.22 / 0.01	4235.11
	3	90062	2438.43	5.70	**89848.00**	0.24 / 0.02	3499.65	2.80	89847.60	0.24 / 0.02	3857.10
	4	89952	1553.53	6.40	89731.60	0.25 / 0.01	3822.29	3.50	**89735.80**	0.24 / 0.02	3946.81
50	1	69984	1626.84	6.30	69773.20	0.30 / 0.02	3504.64	2.90	**69780.00**	0.29 / 0.01	3912.82
	2	70002	2444.74	6.50	69794.00	0.29 / 0.01	3701.30	3.30	**69798.40**	0.29 / 0.02	3977.30
	3	69956	1480.32	6.60	69741.60	0.31 / 0.02	3632.59	3.20	**69749.00**	0.30 / 0.02	3917.03
	4	70064	1473.25	6.50	69850.20	0.31 / 0.01	3812.63	3.30	**69855.80**	0.30 / 0.01	4048.55
60	1	49920	1609.70	7.40	**49715.80**	0.41 / 0.02	3717.88	3.60	49714.40	0.41 / 0.02	3850.08
	2	50094	1926.58	6.40	49879.20	0.42 / 0.02	3562.71	3.20	**49888.80**	0.40 / 0.02	3869.57
	3	50074	2485.99	6.70	49876.40	0.39 / 0.02	3662.16	2.90	**49881.00**	0.39 / 0.02	4035.61
	4	50000	1504.92	6.50	49791.40	0.42 / 0.01	3678.25	3.40	**49804.80**	0.39 / 0.02	4119.61
G	1	6528526722	1673.16	5.20	**6457847194.60**	1.08 / 0.02	3855.91	3.10	6452574600.70	1.16 / 0.03	4104.11
	2	6554507128	1688.89	5.50	**6484463753.50**	1.06 / 0.02	3871.09	3.30	6480230701.70	1.13 / 0.03	4067.86
	3	6548455479	1706.82	5.10	**6476848337.70**	1.09 / 0.03	3938.77	3.10	6473079819.30	1.15 / 0.03	4353.24
	4	6545055776	1659.93	5.20	**6474477296.90**	1.08 / 0.03	3870.45	3.50	6471527989.50	1.12 / 0.05	4297.28

Conclusions

- Over “bivariate” graphs:
  - *Negative subgraph* presents solutions within a relative difference between 0.21 and 0.42.
  - *Complement* slightly improves the solution.
- Over “gaussian” graphs:
  - The results with both algorithms are weaker.
- *Negative subgraph* can be extended any graph topology.

A heuristic for max-cut in toroidal grid graphs

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